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EXTENDING PARTIAL ISOMORPHISMS FOR THE SMALL INDEX PROPERTY OF MANY ω-CATEGORICAL STRUCTURES

BY

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ABSTRACT

THEOREM: *Let A be a finite K_m-free graph, $p_1, \ldots, p_n$ partial isomorphisms on A. Then there exists a finite extension B, which is also a K_m-free graph, and automorphisms f_i of B extending the p_i.*

A paper by Hodges, Hodkinson, Lascar and Shelah shows how this theorem can be used to prove the small index property for the generic countable graph of this class. The same method also works for a certain class of continuum many non-isomorphic ω-categorical countable digraphs and more generally for structures in an arbitrary finite relational language, which are built in a similar fashion. Hrushovski proved this theorem for the class of all finite graphs [Hr]; the proof presented here stems from his proof.

1. Introduction

We say a class $\mathfrak{K}$ of structures has the **extension property** for partial isomorphisms, (EP) for short, if for every finite structure $A \in \mathfrak{K}$ and $p_1, \ldots, p_n$ partial isomorphisms on A, there exists a finite structure $B \in \mathfrak{K}$ and $f_1, \ldots, f_n \in \mathrm{Aut}(B)$ such that f_i extends p_i. The main points are: We want to stay in the realm of finite structures, so B has to be finite; we want to stay inside the class $\mathfrak{K}$; we

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want to solve the extension problem simultaneously in one extending structure for finitely many partial isomorphisms.

The question "does the (EP) hold for graphs" arose in work by Hodges, Hodkinson, Lascar and Shelah on the small index property for the random graph [HHLS]. They realized that this question (EP) is important to understand the automorphism group of the random graph as a topological group. More precisely: The extension property for graphs just shows that most of the n-tuples of automorphisms of the random graph (in the sense of Baire Category) are locally finite. See also Section 5 for this relationship. We want to extend these results to more classes of structures.

In Section 4 of this paper we will consider classes $\mathfrak{K}$ of relational irreflexive structures, which are definable by axioms of a certain type, namely axioms of the type "every structure in $\mathfrak{K}$ weakly avoids T" (where T is a "packed" S-structure; by a packed structure we mean a structure T such that for every $a, b \in T$, if $a \neq b$ then there exists an atomic formula $Ra_1 \cdots a_r$ which holds and in which a and b are appearing) and axioms of the type "if $a_1, \ldots, a_m \in A \in \mathfrak{K}$ are elements which are linked (i.e. $Ra_1 \cdots a_m$ holds for some R in the language), then the quantifier-free formula $\varphi(a_1, \ldots, a_m)$ holds". We will show that if a class has such an axiom system, then the (EP) holds for $\mathfrak{K}$. From this it follows with the same proof as in [HHLS] that the automorphism group of the generic countable structure of this class —which exists in this case— has the small index property.

There are two examples of such classes we want to mention:

1. The K_m-free graphs, where $m \in \omega$ and K_m is the complete graph with m edges.
2. Certain classes of digraphs, which are defined in a similar way by omitting a given (possibly infinite) set of tournaments.

This second example is due to Henson, who introduced these digraphs to construct continuum many non-isomorphic countable ω-categorical digraphs ([Hen]). The results in this paper will show that all the automorphism groups of these digraphs have the small index property. In [Hg] we have already treated the K_3-free case. Also we introduced the technique of permorphisms, which we are also going to use in the present paper. We will need a general extension lemma for permorphisms, which we prove in Section 2. This lemma is nearly already proved in [Hg], but unfortunately not in the generality which is needed here. Furthermore we need a special feature of this extension, which we did not prove there. It might be a good strategy to start reading Section 3 before reading Section 2, because there is a more detailed description of the ideas of the proofs; in particular the notion of permorphisms is motivated there.

In Section 3 we will prove the extension property for Example 1. In Section 4 we will handle the general case. We have chosen to go this way rather than to first prove the general case, because we think that the proof in the general case without an understanding of the simpler case is quite hard to follow. Also Section 3 is meant for the reader who is not interested in arbitrary relational structures, but only in graphs, and who might want to skip the proof in Section 4. For technical reasons we restrict attention to irreflexive structures in Section 4.

In a joint paper with Daniel Lascar [HL] there will be treated a weaker variant of the (EP): a class $\mathfrak{K}$ has the **weak extension property** for partial isomorphisms (WEP), if whenever for $A, p_1, \ldots, p_n$ there exists a possibly infinite solution $M, g_1, \ldots, g_n$, then there exists also a finite solution $B, f_1, \ldots, f_n$ (A, B, M structures in $\mathfrak{K}$; a solution is a structure in $\mathfrak{K}$ together with an n-tuple of automorphisms extending the given partial isomorphisms). There the (WEP) for several classes will be proved and a connection between this property and results on the free groups in finitely many generators considered as topological groups will be developed. In the cases we consider in this paper the (EP) and the (WEP) are equivalent, because there it is easy to give an infinite solution.

In Section 5 we will explain the connection between the (EP) for a class $\mathfrak{K}$ and properties of the automorphism group of the generic countable structure in $\mathfrak{K}$, considered as a topological group. This relationship has already been explained in [HHLS]; we will only add a few remarks due to the slightly different situation.

In Section 6 we are discussing some of the preconditions under which the general theorem was proved in Section 4. In particular we extend the theorem to structures which are not irreflexive.

Apart from the two areas already mentioned (the Small Index Property and topological properties of free groups) there are other fields in which various extension results for partial isomorphisms are used. Let me mention infinite combinatorics (see [GGK]), finite model theory (see [Gr]) and cylindric algebras (see [AHN]). Notably in the last field one uses a strong result on extending partial isomorphisms.

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Notation: By $A \subset B$ we denote inclusion (and not proper inclusion). If p is a partial function on a set A, then $D(p)$ will denote the domain of p, and by $R(p)$

the range of p (so $D(p) \subset A$ and $R(p) \subset A$). If p and f are functions we write $p \subset f$ for f extends p, i.e. $D(p) \subset D(f)$ and for $a \in D(p)$: $a^p = a^f$. By a **partial mapping** we will always mean an injective partial function. If $\bar{a}$ is a tuple, say $\bar{a} = a_1 \cdots a_r$ and D is a set, we write $\bar{a} \in D$ for $a_1 \in D$ and ... and $a_r \in D$, and if $\bar{a} \in D(p)$ we denote by $\bar{a}^p$ the tuple $a_1^p, \dots, a_r^p$. The cardinality of a set D will be denoted by $\#D$. Throughout the paper $p_1, \dots, p_n$ will be partial mappings and $D_i := D(p_i)$ and $D_i' := R(p_i)$, so p_i maps D_i bijectively to D_i'.

2. The plain lemma for permorphisms

In this section we are working in the class of all finite S-structures, for a finite relational language S. "Plain lemma" is just meant to mean: no further restrictions on the S-structures; it is the basic step.

Definition: Let S be a relational language. Let χ be a permutation of the symbols in S mapping every symbol to a symbol with the same arity. Let A be an S-structure and p be a partial mapping on A. We call p a χ-**permorphism**, if for every $r \in \omega$ and every r-ary relation R in S and every $a_1, \dots, a_r \in D(p)$: $Ra_1 \cdots a_r \iff R^\chi a_1^p \cdots a_r^p$.

LEMMA 1: *Let S be a finite relational language, let $\chi_1, \dots, \chi_n$ be permutations of the language mapping every symbol to a symbol of the same arity. Let A be a finite S-structure. Let $p_1, \dots, p_n$ be partial mappings on A, such that p_i is a χ_i-permorphism. Then there exists a finite S-structure B, $A \subset B$, and $f_1, \dots, f_n \in \mathrm{Sym}(B)$ such that each f_i is a χ_i-permorphism and such that $p_i \subset f_i$. We can choose B to satisfy in addition:*

1. *$\forall b \in B \exists f \in \langle f_1, \dots, f_n \rangle$: $b^f \in A$,*
2. *for every R r-ary in S for all $b_1, \dots, b_r \in B$ if $Rb_1 \cdots b_r$ then there exists $f \in \langle f_1, \dots, f_n \rangle$ such that for $1 \le i \le r$, $b_i^f \in A$,*
3. *If the maximal arity of S is bigger than 1 then: whenever there is given $f \in \langle f_1, \dots, f_n \rangle$ and $a, b \in A$ such that $a^f = b$ then there exists $t \in \omega$ $p_{i_1}, \dots, p_{i_t} \in \{p_1, \dots, p_n\}$ and $\epsilon_1, \dots, \epsilon_t \in \{-1, 1\}$ such that $a^{p_{i_1}^{\epsilon_1} \cdots p_{i_t}^{\epsilon_t}} = b$ and $f_{i_1}^{\epsilon_1} \cdots f_{i_t}^{\epsilon_t} = f$.*

$\langle f_1, \dots, f_n \rangle$ is the subgroup of $\mathrm{Sym}(B)$ generated by $f_1, \dots, f_n$. Conditions 1 and 2 just mean that B only contains the structure which is necessary. These conditions are easy to achieve even if B would not have them to begin with: One first replaces B by the substructure $\{a^f \mid a \in A, f \in \langle f_1, \dots, f_n \rangle\}$ and then one cuts down the interpretation of a relation symbol R to all the tuples of the form $\bar{a}^f$, where $\bar{a} \in A$ and $f \in \langle f_1, \dots, f_n \rangle$ and $R\bar{a}$ holds (in A). In Condition 3

$p_{i_1}^{\epsilon_1}\cdots p_{i_t}^{\epsilon_t}$ means the concatenation of partial mappings. In the case $t=0$ this empty concatenation of partial isomorphisms is defined to be the identity on A. The condition means: a and b in the original structure A are only mapped to each other via f inside B if this is necessary given the information of the p_i. We will have this condition automatically at the end. The precondition on the maximal arity in 3 is just included to make the proof easier. We are not interested in the case of just unary relation symbols.

Proof: The proof is by induction on the maximal arity of relation symbols in S. The case of this arity being 1 is easy and is already treated in [Hg].

Let the maximal arity be s, $s\geq 2$. Let S_s be the set of relation symbols in S of arity s.

Definition:

1. A **Δ-type** over A is a collection of formulae of the form $Rx\bar{a}$, where $R\in S$ and $\bar{a}\in A$. If $C\supset A$ and $c\in C$ then by $\Delta\text{-}\mathrm{tp}(c/A)$ we denote the collection of all the formulae of form $Rx\bar{a}$ which hold for c in C. Note that the meaning of "$\Delta\text{-}\mathrm{tp}(c/A)$" will be different in the different chapters.
2. If p is a Δ-type over A and $A_0\subset A\subset C$ and $c\in C$, then $p\restriction A_0 := \{Rx\bar{a}\in p \mid \bar{a}\in A_0\}$ and $\Delta\text{-}\mathrm{tp}(c/A_0) := \Delta\text{-}\mathrm{tp}(c/A)\restriction A_0 = \{Rx\bar{a}\mid \bar{a}\in A_0, Rc\bar{a}\text{ holds}\}$.
3. If φ is a formula of the form $Rx\bar{a}$ with $\bar{a}\in D_i$, then $\varphi^{p_i} := R^{\chi_i}x\bar{a}^{p_i}$; note that φ^{p_i} does not only depend on p_i but also on χ_i.
4. If p is a Δ-type over D_i then $p^{p_i} := \{\varphi^{p_i}\mid \varphi\in p\}$ is a Δ-type over D_i'. It is the type p transported by the permorphism p_i, (remember $D_i = D(p_i), D_i' = R(p_i)$).

CLAIM: *There exists a finite S-structure C, $A\subset C$ such that*

(a) *there exists a constant c_0 such that for every Δ-type p over A:*

$$\#\{c\in C \mid \Delta\text{-}\mathrm{tp}(c/A)=p\}=c_0,$$

(b) *there exists bijections $h_1,\ldots,h_n\in \mathrm{Sym}(C)$, $h_i\supset p_i$ such that for every $\bar{a}\in D_i$, $b\in C$, $R\in S$: $Rb\bar{a}\iff R^{\chi_i}b^{h_i}\bar{a}^{p_i}$.*

Proof: (a) For a Δ-type p over A we let $c_p := \#\{c\in A \mid \Delta\text{-}\mathrm{tp}(c/A)=p\}$ and $c_0 := \max\{c_p \mid p\ \Delta\text{-type over } A\}$. Now for every Δ-type p over A we add (c_0-c_p) many new points c with $\Delta\text{-}\mathrm{tp}(c/A)=p$. Note that we do not consider instances of the relations between these new points; for a clear picture we can suppose that every new instance of a relation involves exactly one new point (and some old points from A).

(b) Fix i ($1 \leq i \leq n$). Note that the statement of (b) just says that if $\Delta\text{-}\mathrm{tp}(b/D_i) = p$, then $\Delta\text{-}\mathrm{tp}(b^{h_i}/D_i') = p^{p_i}$. Let p be a Δ-type over D_i and $p' := p^{p_i}$.

The key for the proof of 2 is the following observation: there are as many possibilities to extend the type p to a type over A as there are to extend p'. More precisely, if we denote $Q(p) = \{q \mid q\ \Delta\text{-type over } A, q\restriction D_i = p\}$ and likewise $Q(p') = \{q \mid q\ \Delta\text{-type over } A, q\restriction D_i' = p'\}$ then $\#Q(p) = \#Q(p')$:

For any $A_0 \subset A$ let $F(A_0) = \{Rx\bar{a} \mid R \in S, \bar{a} \in A_0\}$. Obviously $\#F(D_i) = \#F(D_i')$, so $\#(F(A)-F(D_i)) = \#(F(A)-F(D_i'))$. The elements of $F(A)-F(D_i)$ are the formulae we might add to p without changing the type over D_i: $Q(p) = \{p \cup r \mid r \subset (F(A) - F(D_i))\}$.

So $\#Q(p) = 2^{\#(F(A)-F(D_i))}$ and likewise $\#Q(p') = 2^{\#(F(A)-F(D_i'))} = \#Q(p)$.

Now we prove the crucial equality

$$\#(\{c{\in}C \mid \Delta\text{-}\mathrm{tp}(c/D_i) = p\}) = \#(\{c{\in}C \mid \Delta\text{-}\mathrm{tp}(c/D_i') = p^{p_i}\}):$$

$$\#(\{c{\in}C \mid \Delta\text{-}\mathrm{tp}(c/D_i) = p\}) = \sum_{q \in Q(p)} \#(\{c{\in}C \mid \Delta\text{-}\mathrm{tp}(c/A) = q\}) = c_0 \cdot \#Q(p)$$

and likewise

$$\#(\{c{\in}C \mid \Delta\text{-}\mathrm{tp}(c/D_i') = p'\}) = c_0 \cdot \#Q(p') = c_0 \cdot \#Q(p).$$

Now we can choose h_i to map every set $\{c{\in}C \mid \Delta\text{-}\mathrm{tp}(c/D_i) = p\}$ bijectively to $\{c{\in}C \mid \Delta\text{-}\mathrm{tp}(c/D_i') = p^{p_i}\}$; furthermore we can choose h_i to extend p_i because if $a{\in}D_i$ and $a{\in}\{c{\in}C \mid \Delta\text{-}\mathrm{tp}(c/D_i) = p\}$, then $a^{p_i}{\in}\{c{\in}C \mid \Delta\text{-}\mathrm{tp}(c/D_i') = p^{p_i}\}$.

We introduce the language $S' := (S - S_s) \cup \{R_c \mid c{\in}C, R{\in}S_s\}$, where the R_c are new $(s-1)$-ary relation symbols. We consider A as an S'-structure (and call it A') by defining

$$A' \models R_c\bar{a} \Longleftrightarrow C \models Rc\bar{a}.$$

The maximal arity in S' is $(s-1)$. If we define χ_i' on $\{R_c \mid c{\in}C, R{\in}S_s\}$ by $R_c^{\chi_i'} := (R^{\chi_i})_{c^{h_i}}$ (and for $R{\in}S - S_s$ by $R^{\chi_i'} := R^{\chi_i}$), then p_i is a χ_i'-permorphism on A'. So we get by induction a finite S'-structure F, $A' \subset F$ and $\varphi_1, \ldots, \varphi_n \in \mathrm{Sym}(F)$, $\varphi_i \supset p_i$, φ_i a χ_i'-permorphism. We will not need the extra properties for F mentioned in the lemma.

Now we let $\Gamma \subset \mathrm{Sym}(S) \times \mathrm{Sym}(F) \times \mathrm{Sym}(C)$ be the subgroup generated by the elements $\gamma_i := (\chi_i, \varphi_i, h_i)$. As a convention the components of an element $\gamma{\in}\Gamma$ will always be called χ, φ and h, i.e. $\gamma = (\chi, \varphi, h)$. Γ operates on S (by $R^\gamma := R^\chi$), on F (by $d^\gamma := d^\varphi$), on C (by $c^\gamma := c^h$), and on S' (by $(R_c)^\gamma := (R^\chi)_{c^h}$). But we have to be careful with this notation, because for $a \in A$, a^γ might mean a^h or a^φ, which might be different; we will only apply the notation to relation symbols. For $\gamma = (\chi, \varphi, h) \in \Gamma$ we have that φ is a γ-permorphism on F (of course only with respect to relation symbols in S'). This is true for the generators of Γ and it extends to the whole group.

On $A \times \Gamma$ we define the equivalence relation $\equiv$ to be the symmetric, reflexive and transitive closure of

$$E = \{((a^{p_i}, \gamma), (a, \gamma_i\gamma)) \mid 1 \leq i \leq n, a \in D_i, \gamma \in \Gamma\}.$$

We note some basic facts:

(1) If $(a, (\chi, \varphi, h)) \equiv (a^*, (\chi^*, \varphi^*, h^*))$ then $a^\varphi = (a^*)^{\varphi^*}$ and $a^h = (a^*)^{h^*}$.

(2) If $R \in S \cup S'$ is r-ary and $(a_1, \gamma) \equiv (a_1^*, \gamma^*)$ and ... and $(a_r, \gamma) \equiv (a_r^*, \gamma^*)$ then $R^{\gamma^{-1}} a_1 \cdots a_r \iff R^{(\gamma^*)^{-1}} a_1^* \cdots a_r^*$.

Proof:

(1) It suffices to prove (1) in the cases $((a, (\chi, \varphi, h)), (a^*, (\chi^*, \varphi^*, h^*))) \in E$, $(a^*)^{p_i} = a$ and $(\chi^*, \varphi^*, h^*) = (\chi_i, \varphi_i, h_i) \cdot (\chi, \varphi, h)$. But then $a^\varphi = (a^*)^{p_i \varphi} = (a^*)^{\varphi_i \varphi} = (a^*)^{\varphi^*}$ and $a^h = (a^*)^{p_i h} = (a^*)^{h^*}$.

(2) Let $\gamma = (\chi, \varphi, h)$ and $\gamma^* = (\chi^*, \varphi^*, h^*)$, so φ is a γ-permorphism and φ^* is a γ^*-permorphism (on the S'-structure F).

CASE S': We consider A'. We have for $R \in S'$: $R^{\gamma^{-1}} a_1 \cdots a_r \iff R a_1^\varphi \cdots a_r^\varphi \iff R (a_1^*)^{\varphi^*} \cdots (a_r^*)^{\varphi^*} \iff R^{(\gamma^*)^{-1}} a_1^* \cdots a_r^*$. For the middle equivalence we used (1).

CASE S: Let $R \in S_s$ (otherwise it follows from the previous case).

$R^{\chi^{-1}} a_1 \cdots a_s \iff (R^{\chi^{-1}})_{a_1} a_2 \cdots a_s \iff R_{a_1^h} a_2^\varphi \cdots a_s^\varphi \iff$

$\iff R_{(a_1^*)^{h^*}} (a_2^*)^{\varphi^*} \cdots (a_s^*)^{\varphi^*} \iff R^{(\chi^*)^{-1}} a_1^* \cdots a_s^*$. Here we are using that φ is a γ-permorphism for S' and that $((R^{\chi^{-1}})_{a_1})^\gamma = R_{a_1^h}$ and similarly with the $*$.

Now we are ready to define an S-structure on $A \times \Gamma / \equiv$:

For $e_1, \ldots, e_r \in A \times \Gamma / \equiv$ and $R \in S$ we define:

$$R e_1 \ldots e_r \iff \exists \gamma \in \Gamma \exists a_1, \ldots, a_r \in A (e_i = (a_i, \gamma)/\equiv \text{ and } R^{\gamma^{-1}} a_1 \ldots a_r).$$

We note that

(3) $R^\gamma (a_1, \gamma)/\equiv \cdots (a_r, \gamma)/\equiv \iff R a_1 \cdots a_r$.

(4) If $(a, \gamma_1) \equiv (b, \gamma_2)$, then there exists $t \in \omega$, $p_{i_1}, \ldots, p_{i_t} \in \{p_1, \ldots, p_n\}$ and $\epsilon_1, \ldots, \epsilon_t \in \{-1, 1\}$ such that $b^{p_{i_1}^{\epsilon_1} \cdots p_{i_t}^{\epsilon_t}} = a$ and $\gamma_2 = \gamma_{i_1}^{\epsilon_1} \cdots \gamma_{i_t}^{\epsilon_t} \gamma_1$.

(3) follows directly from (2) and it suffices to prove (4) in the case $(a, \gamma_1) E (b, \gamma_2)$ and $a = b^{p_i}$, $\gamma_2 = \gamma_i \gamma_1$, where it is obvious. In fact in (4) also the reverse

implication holds, so (4) just gives a description of the reflexive, symmetric and transitive closure of E.

We define a map i: $A \to A \times \Gamma/\equiv$ by $i(a) := (a,1)/\equiv$ (where 1 is the unit element in the group). i is injective: if $(a,1) \equiv (b,1)$ then, by (1), $a = b$. By (3), i is an embedding of A into $A \times \Gamma/\equiv$ as S-structures. By identifying we suppose $A \subset A \times \Gamma/\equiv$ and we set $B = A \times \Gamma/\equiv$.

We define the permorphism f_i (for $1 \le i \le n$) by $((a,\gamma)/\equiv)^{f_i} := (a, \gamma\gamma_i)/\equiv$. f_i is well defined; f_i is an χ_i-permorphism, because of fact (3); for $1 \le i \le n$ we have $p_i \subset f_i$, because if $a^{p_i} = b$ then $((a,1)/\equiv)^{f_i} = (a,\gamma_i)/\equiv = (a^{p_i},1)/\equiv = (b,1)/\equiv$.

Properties 1 and 2 are easy; for the additional property 3 suppose $((a,1)/\equiv)^f = (b,1)/\equiv$, where $f \in \langle f_1,\ldots,f_n\rangle$. Take $\gamma \in \Gamma$ such that for all $d \in A$ and $\gamma' \in \Gamma$ $((d,\gamma')/\equiv)^f = (d,\gamma'\gamma)/\equiv$. Then $(b,1) \equiv (a,\gamma)$ and by fact (4) there exists $p_{i_1},\ldots,p_{i_t} \in \{p_1,\ldots,p_n\}$ and $\epsilon_1,\ldots,\epsilon_t \in \{-1,1\}$ such that $a^{p_{i_1}^{\epsilon_1}\cdots p_{i_t}^{\epsilon_t}} = b$ and $\gamma = \gamma_{i_1}^{\epsilon_1}\cdots\gamma_{i_t}^{\epsilon_t}$. So $f = f_{i_1}^{\epsilon_1}\cdots f_{i_t}^{\epsilon_t}$ as desired. ∎

The size of the structure B constructed in this proof is very large in terms of the given data. Therefore we did not try hard to be a little more economical in our construction of the structure B. But nevertheless we want to give an upper bound for the size of B. This bound is an iterated exponential function. We denote by $\mathrm{itexp}(r,\cdot)$ the r times iterated exponential function on natural numbers, which has the following inductive definition: $\mathrm{itexp}(0,s) := s$; $\mathrm{itexp}(r+1,s) := 2^{\mathrm{itexp}(r,s)}$. Let us suppose the maximal arity of relation symbols in S is $r \ge 2$ and that S contains l_j many symbols of arity j (for $1 \le j \le r$). Let $p(x) = l_1 + l_2x + \cdots + l_{r-1}x^{r-2} + 2l_rx^{r-1}$.

Claim: *If we suppose the structure A is of cardinality c, then for the structure B constructed in the proof of Lemma 1 we have $\#B \le \mathrm{itexp}(2r-1, p(c))$.*

The proof of this claim is fairly routine but a little bit tedious. We will only sketch the proof. To start one has to compute that in the unary case one can find a structure B of size $c \cdot 2^{l_1}$. Then the proof is by induction on r. We assume $r \ge 3$ (the case $r = 2$ being similar). The number of Δ-types over A equals $t = 2^{p(c)-l_rc^{(r-1)}}$. The cardinality of C constructed at the beginning of the proof can be bounded by ct. The number of new $(r-1)$-ary predicates in the new language S' is bounded by $l_r \cdot c \cdot t$. By induction we find a structure F of size bounded by $\mathrm{itexp}(2r-3, l_1 + \cdots + l_{r-2}c^{r-3} + 2l_{r-1}c^{r-2} + 2l_rc^{r-1}t)$. For $\mathrm{Sym}(F)$ we are using the bound $2^{\log_2\#F\cdot\#F}$. We obtain a bound for $A \times \Gamma$ by multiplying c and a bound for the number of arity preserving permutations of S and a bound for $\mathrm{Sym}(C)$ and for $\mathrm{Sym}(F)$. As the other numbers are fairly small compared to

$\mathrm{Sym}(F)$, you can easily obtain the bound $\mathrm{itexp}(2r-1, p(c))$ for this number (the additional term $l_r c^{r-1}$ is just meant to swallow all these smaller numbers). As $B = A \times \Gamma/_\equiv$ the above bound also holds for B.

3. The case of the graphs

Definition: In this paper graphs are undirected and loop free and R denotes the edge relation.

A graph A is called K_m-**free** (for $m \in \omega$), if K_m, which is the complete graph with m vertices, is not embeddable into A, i.e. there do not exist $a_1, \ldots, a_m \in A$ such that $a_k R a_l$ (for $1 \leq k < l \leq m$).

THEOREM 2: *Let $m, n \geq 1$, let A be a finite K_m-free graph. Let $p_1, \ldots, p_n$ be partial isomorphisms on A. There exists a finite K_m-free graph $B, B \supset A$, and $f_1, \ldots, f_n \in \mathrm{Aut}(B)$, such that $f_i \supset p_i$.*

Before going through the formal proof, which looks a little bit technical, let us describe the main ideas of the proof:

The proof in the general case will be by induction on m. Let us introduce for every $a \in A$ a new colour (i.e. a unary predicate) U_a such that if $b \in A$ then b is of colour U_a iff bRa. Now A is K_m-free if and only if you cannot embed K_{m-1} "uni-coloured" into A; the latter means there does not exist a colour U_a and elements $a_1, \ldots, a_{m-1}$ in A such that $a_k R a_l$ $(1 \leq k < l \leq m)$ and all the a_i are of colour U_a. Thus one can reduce K_m-freeness conditions to certain K_{m-1}-freeness conditions, if one works with coloured graphs. Here are the main problems which one has to overcome doing this reduction.

1. With respect to the colours, p_i is no longer a partial isomorphism, but it is a permorphism, i.e. it respects the colours only up to a permutation χ_i of the colours $((U_a)^{\chi_i} = U_{a^{p_i}})$.

2. χ_i is not yet really a permutation of the set of colours $\{U_a \mid a \in A\}$, but it is only a partial function. As in [Hg] one overcomes this problem by doing a type-realizing step to get a nice graph $C \supset A$; afterwards one looks at the colours $\{U_d \mid d \in C\}$ and extends the partially already defined functions χ_i to permutations of this set.

3. If one extends the graph A considered as a (uni-coloured K_{m-1})-free graph to a (uni-coloured K_{m-1})-free graph B and the p_i to f_i, how can one ensure that B is K_m-free? Take into account that B is $\{U_d \mid d \in C\}$-coloured, so

we don't have for every $b \in B$ a colour U_b such that all neighbours of b have colour U_b. This problem disappears by a miracle: The resulting graph B looks locally like A. In particular, for every $a \in A$ the neighbours of a in B will still all have colour U_a. Now any orbit of the automorphism group of B (as graph) will have an element inside A. So to check K_m-freeness of B, one has only to look for copies of K_m having one element a inside A, but such a copy would lead to a copy of K_{m-1} of colour U_a.

4. To get a proper induction, one has to prove the theorem for coloured graphs and permorphisms. Starting with a coloured graph, we have to introduce a new set of colours $\{U_d \mid d \in C\}$. But (uni-coloured K_m)-freeness does not exactly mean (uni-coloured K_{m-1})-freeness with respect to the new colours. It means precisely: there do not exist an old colour U, some a of colour U, a new colour U_a and a U_a-coloured copy of K_{m-1} which is at the same time U-coloured. We will call such a combination (U, U_a) a critical colouring, and we have to avoid copies of K_{m-1} which are critical coloured. This last problem and the notational complication arising from the fact that we are dealing with permorphisms rather than isomorphisms will make the proof look rather technical.

The definitions which follow, and the version of the theorem (i.e. Lemma 3), which will be provable by induction, should now be sufficiently motivated.

Definition: Let $\mathfrak{U}^1, \ldots, \mathfrak{U}^r$ be a family of disjoint finite sets of unary predicates (called **colours**). Let $\mathfrak{U} := \bigcup_{1 \leq j \leq r} \mathfrak{U}^j$. If A is an $\{R\} \cup \mathfrak{U}$-structure and $a \in A$, $V \in \mathfrak{U}$, then we write $a \in V$ or Va to indicate that the unary predicate V (or rather its interpretation in A) is true for a; we also write "a is of colour V".

1. A **$\mathfrak{U}$-graph** is an $\{R\} \cup \mathfrak{U}$-structure A such that A considered as an $\{R\}$-structure is a graph.

   For $a \in A$ we define $\mathfrak{U}(a) := \{V \in \mathfrak{U} \mid a \in V\}$.

2. Let $\mathfrak{U}_c \subset \mathfrak{U}^1 \times \cdots \times \mathfrak{U}^r$. We will call $\mathfrak{U}_c$ the set of **critical colourings**. We say that A is **$\mathfrak{U}_c$-K_m-free** (for $m \in \omega$), if there do not exist a colouring $(V_1, \ldots, V_r) \in \mathfrak{U}_c$ and elements $a_1, \ldots, a_m \in A$, such that $a_k R a_l$ (for $1 \leq k < l \leq m$) and $a_k \in V_j$ (for $1 \leq k \leq m, 1 \leq j \leq r$).

The next lemma is the permorphism version of Theorem 2. This is the version we are able to prove by induction on m.

LEMMA 3: *Let $m \geq 1$. Let $\mathfrak{U}^1, \ldots, \mathfrak{U}^r$ be disjoint sets of colours (where $r \geq 0$). Let $\mathfrak{U} := \bigcup_{1 \leq j \leq r} \mathfrak{U}^j$. Let $\chi_i^j \in \mathrm{Sym}(\mathfrak{U}^j)$ (for $1 \leq i \leq n, 1 \leq j \leq r$), $S := \mathfrak{U} \cup \{R\}$ $\chi_i := \bigcup_{0 \leq j \leq r} \chi_i^j \in \mathrm{Sym}(S)$, where χ_i^0 is the identity on $\{R\}$. Furthermore let $\mathfrak{U}_c \subset \mathfrak{U}^1 \times \cdots \times \mathfrak{U}^r$ be a set called critical colourings. Suppose $\mathfrak{U}_c$ is χ_i-invariant (for $1 \leq i \leq n$).*

Let A be a finite $\mathfrak{U}_c$-K_m-free $\mathfrak{U}$-graph and suppose $p_1, \ldots, p_n$ are partial mappings on A such that p_i is a χ_i-permorphism.

Then there exist a finite $\mathfrak{U}$-graph B, $B \supset A$, B $\mathfrak{U}_c$-K_m-free, $f_1, \ldots, f_n \in \mathrm{Sym}(B)$, $f_i \supset p_i$, f_i a χ_i-permorphism (for $1 \leq i \leq n$).

B can be chosen to satisfy in addition:

1. *$\forall b \in B \exists f \in \langle f_1, \ldots, f_n \rangle$: $b^f \in A$,*
2. *for all $a, b \in B$ if aRb then there exists $f \in \langle f_1, \ldots, f_n \rangle$ with $a^f, b^f \in A$,*
3. *whenever there exists $f \in \langle f_1, \ldots, f_n \rangle$ and $a, b \in A$ such that $a^f = b$ then there exists $t \in \omega$ $p_{i_1}, \ldots, p_{i_t} \in \{p_1, \ldots, p_n\}$ and $\epsilon_1, \ldots, \epsilon_t \in \{-1, 1\}$ such that $a^{p_{i_1}^{\epsilon_1} \cdots p_{i_t}^{\epsilon_t}} = b$ and $f_{i_1}^{\epsilon_1} \cdots f_{i_t}^{\epsilon_t} = f$.*

From the lemma follows the theorem:

Let $r = 0$, which means we just talk about (uncoloured) graphs. $\mathfrak{U}_1 \times \cdots \times \mathfrak{U}_r$ just contains the empty tuple λ and we let $\mathfrak{U}_c = \{\lambda\}$; then $\mathfrak{U}_c$-K_m-freeness just means K_m-freeness. Now χ_i is the identity on $\{R\}$ and χ_i-permorphism just means isomorphism of graphs.

Proof of the lemma: The proof goes by induction on m. Let us first treat the case $m = 1$.

By Lemma 1 there exists $B \supset A$, B a $\mathfrak{U}$-graph, and $f_1, \ldots, f_n \in \mathrm{Aut}(B)$, $p_i \subset f_i$, f_i a χ_i-permorphism, having all the extra properties we want for B (use property 2 to check that B is really a graph, i.e. R is irreflexive and symmetric). We have to check that B is $\mathfrak{U}_c$-K_1-free. Suppose there exists $b \in B$ and $(V_1, \ldots, V_r) \in \mathfrak{U}_c$, such that $b \in V_j$ $(1 \leq j \leq r)$. Choose $f \in \langle f_1, \ldots, f_n \rangle$ such that $b^f \in A$ and choose $\chi \in \langle \chi_1, \ldots, \chi_n \rangle$ such that f is a χ-permorphism. It follows that $b^f \in A$, $(V_1^\chi, \ldots, V_r^\chi) \in \mathfrak{U}_c, b^f \in V_j^\chi$. This contradicts the $\mathfrak{U}_c$-K_1-freeness of A.

Now we do the step of induction $m \to m+1$ $(m \geq 1)$:

We have the set of colours $\mathfrak{U}^1, \ldots, \mathfrak{U}^r$ and we suppose A to be $\mathfrak{U}_c$-K_{m+1}-free. By a type-realizing step and by introducing new colours we want to consider A as satisfying a certain K_m-freeness condition and then we want to apply the lemma for m.

A subset $A_0 \subset A$ and a colouring $\mathfrak{U}_0 \subset \mathfrak{U}$ determine a type over A in this context, namely the type $\{xRa \mid a \in A_0\} \cup \{Vx \mid V \in \mathfrak{U}_0\}$ (see the following definition). But not all of the types are realizable in $\mathfrak{U}_c$-K_{m+1}-free graphs.

Definition: Let $A \subset C$ be $\mathfrak{U}$-graphs. p will denote a Δ-type over A, c an element of C.

1. A Δ-type over A is a collection of formulae of the form Vx (where $V \in \mathfrak{U}$) and of the form xRa (where $a \in A$). We let

$$\Delta\text{-}\operatorname{tp}(c/A) := \{Vx \mid C \models Vc\} \cup \{xRa \mid a \in A, C \models cRa\}$$

and

$$\operatorname{Par}(p) := \{a \in A \mid xRa \in p\},$$

the parameters of p, and

$$\mathfrak{U}(p) := \{V \in \mathfrak{U} \mid Vx \in p\},$$

the colours of p.

2. We say $c \models p$, if for every formula $\varphi(x) \in p$ we have $\varphi(c)$. Note the difference between $\Delta\text{-}\operatorname{tp}(c/A) = p$ and $c \models p$: $c \models p$ iff $\Delta\text{-}\operatorname{tp}(c/A) \supset p$.

3. We call a Δ-type p over A **realizable**, if there do not exist $(V_1, \dots, V_r) \in \mathfrak{U}_c \cap (\mathfrak{U}(p))^r$ and elements $a_1, \dots, a_m \in \operatorname{Par}(p)$ such that $a_k R a_l$ (for $1 \le k < l \le m$) and $a_k \in V_j$ (for $1 \le k \le m$, $1 \le j \le r$). The intended meaning is: p is realizable if one can realize it in a $\mathfrak{U}_c$-K_{m+1}-free graph.

Here are facts about realizability:

(1) If $p \subset q$ and q is realizable, then so is p.

(2) If $C \supset A$ (C a $\mathfrak{U}$-graph) is $\mathfrak{U}_c$-K_{m+1}-free, then for every $c \in C$, $\Delta\text{-}\operatorname{tp}(c/A)$ is realizable. In particular, for every $a \in A$, $\Delta\text{-}\operatorname{tp}(a/A)$ is realizable.

(3) If p is a Δ-type over A and $\operatorname{Par}(p) \subset D_i$ for some i with $1 \le i \le n$ then we can define $p^{p_i} := \{V^{\chi_i} x \mid Vx \in p\} \cup \{xRa^{p_i} \mid xRa \in p\}$ and we have:

$$p \text{ is realizable} \iff p^{p_i} \text{ is realizable.}$$

All these facts follow directly from the definition of realizability. For (3) one has to use in addition, that $\mathfrak{U}_c$ is χ_i-invariant.

Now we do the type-realizing step:

CLAIM:

(a) *There exists a $\mathfrak{U}_c$-K_{m+1}-free $\mathfrak{U}$-graph $C \supset A$ and for every $t (0 \le t \le \#A)$ a constant c_t such that for every Δ-type p over A:*

$$\#\{c \in C \mid c \models p, \mathfrak{U}(c) = \mathfrak{U}(p)\} = \begin{cases} c_{\#\operatorname{Par}(p)} & \textit{if } p \textit{ is realizable,} \\ 0 & \textit{otherwise.} \end{cases}$$

(b) *There exist bijections $h_1, \ldots, h_n \in \mathrm{Sym}(C)$, $h_i \supset p_i$ such that for every $V \in \mathfrak{U}$, for every $b \in C$, for every i $(1 \leq i \leq n)$: $b \in V \iff b^{h_i} \in V^{\chi_i}$ and for every $a \in D_i$, $b \in C$: $aRb \iff a^{p_i} R b^{h_i}$.*

Proof of (a): Let $T = \#A$. We construct graphs $A = C_T \subset C_{T-1} \subset \cdots \subset C_0 = C$ and constants $c_T, \ldots, c_0$ such that for every t $(T \geq t \geq 0)$ and for every Δ-type p over A with $\#\mathrm{Par}(p) \geq t$:

$$\#\{c \in C_t \mid c \models p, \mathfrak{U}(c) = \mathfrak{U}(p)\} = c_{\#\,\mathrm{Par}(p)} \text{ if } p \text{ is realizable}$$

and such that C_t is $\mathfrak{U}_c$-K_{m+1}-free. We let $c_T = 0$.

If c_r, C_r are already constructed (for $T \geq r \geq t$) and $t \geq 1$ then we will construct C_{t-1} by adding points which have exactly $t-1$ neighbours, all of them in A: For p a realizable Δ-type over A with $\#\,\mathrm{Par}(p) = t-1$ we define $c_p = \#\{c \in C_t \mid c \models p, \mathfrak{U}(c) = \mathfrak{U}(p)\}$ and we define c_{t-1} to be the maximum of all these c_p. Now to get C_{t-1} we add for every realizable p (with $\#\,\mathrm{Par}(p) = t-1$) $c_{t-1} - c_p$ many points having exactly type p (by this we mean points c such that $\Delta\text{-}\mathrm{tp}(c/A) = p$; note that all the neighbours of c are in A). Now C_{t-1} is a $\mathfrak{U}$-graph and $\mathfrak{U}_c$-K_{m+1}-free and for every Δ-type p over A (with $t-1 \leq \#\,\mathrm{Par}(p)$): $\#\{c \in C_{t-1} \mid c \models p, \mathfrak{U}(c) = \mathfrak{U}(p)\} = c_{\#\,\mathrm{Par}(p)}$ if p is realizable. This is true, because if $\mathrm{Par}(p) \geq t$ we did not change the set in question (by going from C_t to C_{t-1}) and if $\#\,\mathrm{Par}(p) = t-1$ then $\#\{c \in C_{t-1} \mid c \models p, \mathfrak{U}(c) = \mathfrak{U}(p)\} = \#\{c \in C_t \mid c \models p, \mathfrak{U}(c) = \mathfrak{U}(p)\} + (c_{t-1} - c_p) = c_p + c_{t-1} - c_p = c_{t-1}$.

Proof of (b): This is similar to the proof of (b) in Lemma 1.

Here again it is crucial to check that for every Δ-type p over D_i:

$$\#\{c \in C \mid \Delta\text{-}\mathrm{tp}(c/D_i) = p\} = \#\{c \in C \mid \Delta\text{-}\mathrm{tp}(c/D_i') = p^{p_i}\}.$$

This is done by downwards induction on the size of $\mathrm{Par}(p)$. We do the induction step in the case p is realizable. Otherwise both sets are empty; here we are using fact (3): p is realizable $\iff p^{p_i}$ is realizable.

$$\begin{aligned}
& \#\{c \in C \mid \Delta\text{-}\mathrm{tp}(c/D_i) = p\} \\
= \quad & \#\{c \in C \mid \Delta\text{-}\mathrm{tp}(c/D_i) \supset p, \mathfrak{U}(c) = \mathfrak{U}(p)\} - \sum_{q \in \mathrm{Ext}(p)} \#\{c \in C \mid \Delta\text{-}\mathrm{tp}(c/D_i) = q\} \\
= \quad & c_{\#\,\mathrm{Par}(p)} - \sum_{q \in \mathrm{Ext}(p)} \#\{c \in C \mid \Delta\text{-}\mathrm{tp}(c/D_i') = q^{p_i}\} \\
= \quad & \#\{c \in C \mid \Delta\text{-}\mathrm{tp}(c/D_i') \supset p^{p_i}, \mathfrak{U}(c) = \mathfrak{U}(p^{p_i})\}
\end{aligned}$$

$$- \sum_{q'\epsilon \mathrm{Ext}(p^{p_i})} \#\{c\in C \mid \Delta\text{-}\mathrm{tp}(c/D_i') = q'\}$$
$$= \#\{c\in C \mid \Delta\text{-}\mathrm{tp}(c/D_i') = p^{p_i}\}$$

Here $\mathrm{Ext}(p) := \{q \mid q\ \Delta\text{-type over } D_i, p \not\subseteq q, \mathfrak{U}(q) = \mathfrak{U}(p)\}$ and similar $\mathrm{Ext}(p^{p_i})$.

Now we introduce a new set of colours: $\mathfrak{U}^{r+1} = \{U_d^{r+1} \mid d\in C\}$, where U_d^{r+1} is a new unary predicate for every $d\in C$. We define $\chi_i^{r+1}\in\mathrm{Sym}(\mathfrak{U}^{r+1})$ (for $1\leq i\leq n$) by $(U_d^{r+1})^{\chi_i^{r+1}} := U_{d^{h_i}}^{r+1}$. We let $\mathfrak{U}' := \mathfrak{U}\cup\mathfrak{U}^{r+1}$ and $\chi_i' := \chi_i \cup \chi_i^{r+1}\in\mathrm{Sym}(\mathfrak{U}'\cup\{R\})$. We define $\mathfrak{U}_c' \subset \mathfrak{U}^1\times\cdots\times\mathfrak{U}^{r+1}$ by

$$(V_1,\ldots,V_r,U_d^{r+1})\in\mathfrak{U}_c' \iff (V_1,\ldots,V_r)\in\mathfrak{U}_c \quad\text{and}\quad d\in V_j \quad (1\leq j\leq r)\ (\text{in } C).$$

$\mathfrak{U}_c'$ is χ_i'-invariant, because $\mathfrak{U}_c$ is χ_i-invariant, because of the definition of χ_i^{r+1} and because of the property of h_i in claim (b) ($b\in V \iff b^{h_i}\in V^{\chi_i}$).

The colours in $\mathfrak{U}^{r+1}$ are in a natural way interpreted in A: for $a\in A$ and $d\in C$ we define $a\in U_d^{r+1} \iff dRa$ (in C). Now A is a $\mathfrak{U}'$-graph.

A is $\mathfrak{U}_c'$-K_m-free. Otherwise there would exist $a_1,\ldots a_m \in A$ and $(V_1,\ldots,V_r,U_d^{r+1})\in\mathfrak{U}_c'$ such that a_kRa_l (for $1\leq k<l\leq m$) and $a_k\in V_j$ (for $1\leq k\leq m$, $1\leq j\leq r$) and $a_k\in U_d^{r+1}$. But then $(V_1,\ldots,V_r)\in\mathfrak{U}_c$, $d\in V_j$, a_kRd. This means $a_1,\ldots,a_m,d$ contradicts the $\mathfrak{U}_c$-K_{m+1}-freeness of C.

For $1\leq i\leq n$ p_i is a χ_i'-permorphism. By the lemma for m we find a finite $\mathfrak{U}'$-graph B, $B\supset A$, B $\mathfrak{U}_c'$-K_m-free and $f_1,\ldots,f_n\in\mathrm{Sym}(B)$, $f_i\supset p_i$, f_i a χ_i'-permorphism having the indicated properties.

Now we consider B just as a $\mathfrak{U}$-graph. The only thing we still have to check is that B is $\mathfrak{U}_c$-K_{m+1}-free.

Claim:

(1) *If $a\in A$ and $b\in B$ then $aRb\Rightarrow b\in U_a^{r+1}$.*

(2) *B is $\mathfrak{U}_c$-K_{m+1}-free.*

Proof: (1) Because of the extra property 2 of the lemma and as aRb we can choose $f\in\langle f_1,\ldots,f_n\rangle$ such that $a_0 := a^f\in A$ and $b_0 := b^f\in A$. As a_0Rb_0 we have $b_0\in U_{a_0}^{r+1}$. Let $f = f_{i_1}^{\epsilon_1}\cdots f_{i_t}^{\epsilon_t}$ and $a^{p_{i_1}^{\epsilon_1}\cdots p_{i_t}^{\epsilon_t}} = a_0$; here we are using the extra property 3. Now f is a $\chi_{i_1}^{\epsilon_1}\cdots\chi_{i_t}^{\epsilon_t}$-permorphism and we also know $a^{h_{i_1}^{\epsilon_1}\cdots h_{i_t}^{\epsilon_t}} = a_0$ so $(U_a^{r+1})^{\chi_{i_1}^{\epsilon_1}\cdots\chi_{i_t}^{\epsilon_t}} = U_{a_0}^{r+1}$. Thus because $b^f = b_0\in U_{a_0}^{r+1}$ it follows: $b\in U_a^{r+1}$.

(2) Suppose there exist $(V_1,\ldots,V_r)\in\mathfrak{U}_c$ and elements $a_0,\ldots,a_m\in B$ such that a_kRa_l and $a_k\in V_j$. W.l.o.g we suppose $a := a_0\in A$: Otherwise choose $f\in\langle f_1,\ldots,f_n\rangle$, such that $a_0^f\in A$ and choose $\chi'\in\mathrm{Sym}(S')$ such that f is a χ'-permorphism and let $\chi = \chi' \mid_S$; now still $(V_1^\chi,\ldots,V_r^\chi)\in\mathfrak{U}_c$ and $a_k^fRa_l^f$ and

$a_k^f \in V_j^\chi$. We have $(V_1, \ldots, V_r, U_a^{r+1}) \in \mathfrak{U}_c'$ and by (1): $a_k \in U_a^{r+1}$ $(1 \leq k \leq m)$. Thus we get a contradiction to the $\mathfrak{U}_c'$-K_m-freeness of B. ∎

The graphs we constructed to prove Theorem 2 are all of the form $A \times \Gamma/\equiv$ where Γ is a big enough finite group generated by n distinguished elements $\gamma_1, \ldots, \gamma_n$. In fact, to prove (EP) for the K_m-free graphs it is equivalent — given $A, p_1, \ldots, p_n$— to finding a finite extending K_m-free graph B and automorphisms $f_1, \ldots, f_n$ extending the p_i and to find such a group Γ such that $B = A \times \Gamma/\equiv$ is K_m-free, $a \mapsto (a, 1)/\equiv$ gives an embedding and for each i the action of γ_i extends p_i. To see this take an arbitrary solution $B, f_1, \ldots, f_n$, then define $\Gamma = \langle f_1, \ldots, f_n \rangle \subset \mathrm{Aut}(B)$. The mapping ρ: $A \times \Gamma/\equiv \ \to B$ defined by $(a, f)/\equiv \ \mapsto a^f$ is well defined and an embedding i: $K_m \to A \times \Gamma/\equiv$ would lead to an embedding $i\rho$: $K_m \to B$. Thus also $A \times \Gamma/\equiv$ is K_m-free and solves the extension problem. Note that ρ is a weak homomorphism in the sense below.

4. A more general case

In this section we are handling a more general case. The theorem we are going to prove will imply Theorem 2 which we just proved. But we hope that it will be much easier to understand this general case, after having understood the case of the graphs. We will start with some definitions we will need for the statement of the theorem. We will restrict attention here to irreflexive structures, even if this is not essential, as the final chapter will show.

Definition: Let S be a relational language.

1. A structure A is called **irreflexive**, if for every $k \in \omega$, every k-ary R in S and every $a_1, \ldots, a_k$ in A, if $Ra_1 \cdots a_k$ holds, then $a_1, \ldots, a_k$ are pairwise distinct.

2. By S_k $(k \in \omega)$ we denote the set of k-ary relation symbols in S.

3. Let L be an S-structure. L is called a **link structure**, if there exists $a_1, \ldots, a_k \in L$ and $R \in S$ such that $\{a_1, \ldots, a_k\} = L$ and $Ra_1 \cdots a_k$ holds in L or if L has just one element.

4. Let $\mathfrak{L}$ be a set of link structures, A be an S-structure. A has **link type** $\mathfrak{L}$ if for every substructure $L \subset A$: if L is a link structure, then there exist $L' \in \mathfrak{L}$ such that $L \cong L'$.

5. Let T,' A be S-structures, let ρ: $T \to A$ be a function. ρ is called a **weak homomorphism** (notation: ρ: $T \to_w A$) if for every $k \in \omega$, $R \in S_k$ and $s_1, \ldots, s_k \in T$: if $Rs_1 \cdots s_k$ (in T) then $Rs_1^\rho \cdots s_k^\rho$ (in A).

6. Let $\mathfrak{F}$ be a set of finite S-structures. Let A be an S-structure. We say A is **$\mathfrak{F}$-free** or A weakly avoids $\mathfrak{F}$, if there does not exist $T \in \mathfrak{F}$ and $\rho: T \to_W A$.

7. Suppose $\mathfrak{L}$ is a set of link structures and $\mathfrak{F}$ is a set of finite S-structures, then $\mathfrak{K}_{\mathfrak{L}\mathfrak{F}}$ will be the class of all finite S-structures A, which are $\mathfrak{F}$-free and of link type $\mathfrak{L}$ and $\mathfrak{K}_{\mathfrak{F}}$ will denote all irreflexive S-structures, which are $\mathfrak{F}$-free.

8. T is a **packed** structure, if for every $t_1, t_2 \in T$ there exists a link-structure $L \subset T$ with $t_1, t_2 \in L$.

Examples: 1. If $\mathfrak{L}$ contains for every irreflexive link structure an isomorphic copy, then for every S-structure B: B is irreflexive iff B has link type $\mathfrak{L}$.

2. Let R be binary, $S = \{R\}$. If we let $\mathfrak{L}_1 = \{K_1, K_2\}$, where K_1 is just one point and K_2 is the graph with two points and an edge between them, then B is a graph iff B has link type $\mathfrak{L}_1$.

3. Now we let T_2 be the structure consisting of two points a, b and xRy holds in T_2 iff $x = a$ and $y = b$. B is a **directed graph** iff B is of link type $\mathfrak{L}_2 = \{K_1, T_2\}$.

PROPOSITION 4: *If T is packed and A is irreflexive and $\rho: T \to_W A$, then ρ is injective.*

If $\mathfrak{F}$ is a family of packed structures then $\mathfrak{K}_{\mathfrak{F}}$ (and also $\mathfrak{K}_{\mathfrak{L}\mathfrak{F}}$) has the **free amalgamation property** *(fAP). By this we mean: If $A, B, C \in \mathfrak{K}_{\mathfrak{F}} \cup \{\emptyset\}$, $A \subset B$, $A \subset C$ then $B *_A C \in \mathfrak{K}_{\mathfrak{F}} \cup \{\emptyset\}$.*

Here $B *_A C$ is the **free amalgam** of B and C over A. To define it, first suppose that $C \cap B = A$ (by changing C to an isomorphic copy). Now the underlying domain of $B *_A C$ is just $B \cup C$ and a relation $R\bar{d}$ holds in $B *_A C$ iff it holds in B (and $\bar{d} \in B$) or in C. Note that we allow the common part to be empty. The proof of the proposition is easy.

THEOREM 5: *Let S be a finite relational language. Let $\mathfrak{F}$ be a set of finite irreflexive packed S-structures. Let $\mathfrak{L}$ be a set of irreflexive link structures. Then $\mathfrak{K}_{\mathfrak{L}\mathfrak{F}}$ has (EP), the extension property for partial isomorphisms.*

Definition: Let χ be an arity-preserving permutation of S. For any S-structure T we define T^χ to be the S-structure with the same underlying domain as T, but for $R \in S$ (R k-ary) we define: $Ra_1 \cdots a_k$ holds in T^χ iff $R^{\chi^{-1}} a_1 \cdots a_k$ holds in T. We say that a family $\mathfrak{F}$ of S-structures is **invariant** under χ, if $T \in \mathfrak{F} \iff T^\chi \in \mathfrak{F}$.

Now we formulate the permorphism version of Theorem 5

LEMMA 6: *Let S be a finite relational language and let $\chi_1, \ldots, \chi_n$ be arity preserving permutations of S. Let $\mathfrak{F}$ be a finite family of finite irreflexive packed S-structures invariant under χ_i (for $1 \le i \le n$). Let $A \in \mathfrak{K}_{\mathfrak{F}}$ be finite. Let $p_1, \ldots, p_n$ be partial mappings on A, p_i a χ_i-permorphism.*

There exist a finite S-structure $B \in \mathfrak{K}_{\mathfrak{F}}$, $A \subset B$ and $f_1, \ldots, f_n$ bijections on B, $f_i \supset p_i$, such that f_i is a χ_i-permorphism. B has the additional properties mentioned in Lemma 1.

From the lemma follows the theorem:

Let $A \in \mathfrak{K}_{\mathfrak{L}\mathfrak{F}}$ and $p_1, \ldots, p_n$ be given.

1. Suppose first that $\mathfrak{F}$ is finite. Let each χ_i be the identity and apply the lemma in this situation. We only have to check that the resulting structure B is automatically of link type $\mathfrak{L}$. This follows from the additional property 1 and 2. In fact suppose $L \subset B$ is a link structure. There exists $f \in \langle f_1, \ldots, f_n \rangle$ such that $L^f \subset A$. Because A has link type $\mathfrak{L}$ there exists $L' \in \mathfrak{L}$ such that $L \cong L^f \cong L'$.

2. Suppose now that for every $T \in \mathfrak{F}$ and every substructure $T' \subset T$, T' is packed. Let m be the size of A. Let $\mathfrak{F}_{m+1}$ be a finite set containing for every packed S-structure of size $m+1$ exactly one isomorphic copy. Suppose further that $\mathfrak{F}$ contains only finitely many structures of size less than $m+1$ (by throwing away isomorphic copies). Let $\mathfrak{F}' := \{T \in \mathfrak{F} \mid \#T \le m\} \cup \mathfrak{F}_{m+1}$. Then $A \in \mathfrak{K}'_{\mathfrak{L}\mathfrak{F}}$ and we can apply the first case to get $B \in \mathfrak{K}'_{\mathfrak{L}\mathfrak{F}}$ solving the extension problem. But B is automatically $\mathfrak{F}$-free. It does not contain any packed substructure of size $m+1$ so it does not (weakly) contain any structure $T \in \mathfrak{F}$ with $\#T > m$.

3. We do the general case by a trick. Let R be a new binary predicate and define $S^* := S \cup \{R\}$. For an S-structure C we define C^* to be the expansion of C where for $b, c \in C$: $bRc \iff b \ne c$. Let

$$\mathfrak{F}^* = \{T^* \mid T \in \mathfrak{F}\} \quad \text{and} \quad \mathfrak{L}^* := \{L \subset A^* \mid L \text{ a link structure}\}.$$

By case 2 we find $B \in \mathfrak{K}^*_{\mathfrak{L}^*\mathfrak{F}}$, $A^* \subset B$, and we only have to show that $B \upharpoonright_S \in \mathfrak{K}_{\mathfrak{L}\mathfrak{F}}$.

If $L \subset B$ such that $L \upharpoonright_S$ is a link structure then $L \cong L' \in \mathfrak{L}^*$, so $L \cong L' \subset A^*$. Thus $L \upharpoonright_S \cong L' \upharpoonright_S \subset A$. But A has link type $\mathfrak{L}$ so $L \upharpoonright_S \cong L' \upharpoonright_S \cong L'' \in \mathfrak{L}$.

Suppose now there exists $T \in \mathfrak{F}$ and $\rho: T \to_w B \upharpoonright_S$ (note id: $B \upharpoonright_S \to_w B$). As B is irreflexive ρ is injective. We want to prove that ρ is a map $T^* \to_w B$, to get a contradiction. For this suppose $t_1, t_2 \in T^*$ and Rt_1t_2. We have to show $Rt_1^\rho t_2^\rho$. Because T is packed there is $P \in S$ and $\bar{t} \in T$, t_1, t_2 appear in $\bar{t}$, such that $P\bar{t}$ holds. Thus $P\bar{t}^\rho$ holds in B. Consider the substructure $\{\bar{t}^\rho\}$ of B. It is a link structure. So it is isomorphic to a substructure of A^*. Thus for any two distinct elements t, t' of $\{\bar{t}\}$, Rtt' holds. In particular, as ρ is injective, $Rt_1^\rho t_2^\rho$ holds.

Proof of the lemma: The proof will be similar to the proof for the graphs in Section 3. We only point out the main differences. It goes by induction on the maximal size of structures in $\mathfrak{F}$. The case that this size is one is as easy as the corresponding case for the graphs. So let us suppose that the maximal size of the structures in $\mathfrak{F}$ is $m+1>1$. We need to change the notion of Δ-type slightly:

Definition:

1. A **Δ-type** over A is a collection of formulae of the form $Ra_1 \cdots a_{k'-1}xa_{k'} \cdots a_{k-1}$, where $k \geq 1$ and $R \in S_k$ and $a_1, \ldots, a_{k-1}$ are pairwise distinct elements of A. The notions about Δ-types in Section 2 like $\Delta\text{-}\operatorname{tp}(c/A)$ and $p\restriction_{D_i}$ and p^{p_i} extend naturally to this notion of Δ-type. If p is a Δ-type over A we define $\operatorname{Par}(p) := \{a \in A \mid$ there exists a formula $\varphi \in p$ in which a appears$\}$.
2. Let p be a Δ-type over A and let $A_0 := \operatorname{Par}(p)$. Suppose $A_0 \subset B$. We denote by $B+p$ the structure which we get by adding just one point c_p to B having exactly type p. So $B+p = B \cup \{c_p\}$, $c_p \notin B$, $\Delta\text{-}\operatorname{tp}(c_p/B) = p$. Note that $B+p$ is irreflexive.
3. Let p be a Δ-type, and $A_0 = \operatorname{Par}(p)$. p is called **realizable** iff $A_0 + p \in \mathfrak{K}_{\mathfrak{F}}$.

The facts about realizability in this context:

(0) If p is realizable and $A_0 \subset B$ and $B \in \mathfrak{K}_{\mathfrak{F}}$, then $B + p \in \mathfrak{K}_{\mathfrak{F}}$.

(1) If $p \subset q$ and q is realizable, then so is p.

(2) If $A \subset C \in \mathfrak{K}_{\mathfrak{F}}$, then for every $c \in C$ $\Delta\text{-}\operatorname{tp}(c/A)$ is realizable. In particular, for every $a \in A$ $\Delta\text{-}\operatorname{tp}(a/A)$ is realizable.

(3) If p is a type over A and $\operatorname{Par}(p) \subset D_i$ (for some i with $1 \leq i \leq n$), then

$$p \text{ is realizable } \Longleftrightarrow p^{p_i} \text{ is realizable.}$$

It is here (in Fact (0), which says that realizable types are indeed realizable) where we use that the structures in $\mathfrak{F}$ are packed.

Proof:

(0) It is $A_0 + p \in \mathfrak{K}_{\mathfrak{F}}$ and $B \in \mathfrak{K}_{\mathfrak{F}}$ so by (fAP) $B + p \cong (A_0 + p) *_{A_0} B \in \mathfrak{K}_{\mathfrak{F}}$.
(1) Note that if $p \subset q$ then $\operatorname{Par}(p) + p \to_w \operatorname{Par}(q) + q$.
(2) Use that for $p = \Delta\text{-}\operatorname{tp}(c/A)$: $A + p \to_w C$.
(3) Observe that $(\operatorname{Par}(p)+p)^{\chi_i} \cong \operatorname{Par}(p^{p_i}) + p^{p_i}$ and use that $\mathfrak{F}$ is χ_i-invariant.

Now the crucial claim in this proof reads as follows:

CLAIM:

(a) *There exists an $\mathfrak{F}$-free S-structure C, $A \subset C$, and for every t a constant $c_t \in \omega$ such that for every $E \subset A$ and every Δ-type p over A such that* $\mathrm{Par}(p) = E$*:*

$$\#\{c \in C \mid \Delta\text{-}\mathrm{tp}(c/E) = p\} = \begin{cases} c_{\#E} & \text{if } p \text{ is realizable,} \\ 0 & \text{otherwise.} \end{cases}$$

(b) *There exist bijections $h_1, \ldots, h_n \in \mathrm{Sym}(C)$, $h_i \supset p_i$ such that for every i $(1 \le i \le n)$ and every $b \in C$: $\Delta\text{-}\mathrm{tp}(b^{h_i}/D_i') = (\Delta\text{-}\mathrm{tp}(b/D_i))^{p_i}$.*

The proof of (a) is as in the case of the graphs. It works because adding a new point of exactly type p (where $\mathrm{Par}(p) = E$) does not change the number of points d with $\Delta\text{-}\mathrm{tp}(d/E') = p'$ for any type p' with $\mathrm{Par}(p') = E'$, provided $\#E' \ge \#E$ and $p' \ne p$. So as in Section 3 we can work our way downwards. Here we use also fact 0; the fact insures that the constructed structure is still $\mathfrak{F}$-free.

Also, the proof of (b) goes exactly along the lines of Section 3. If p is a Δ-type over D_i and $\mathrm{Par}(p) = E$, then

$$\{c \mid \Delta\text{-}\mathrm{tp}(c/D_i) = p\} = \{c \mid \Delta\text{-}\mathrm{tp}(c/E) = p\} - \bigcup_{q \in \mathrm{Ext}(p)} \{c \mid \Delta\text{-}\mathrm{tp}(c/D_i) = q\},$$

where $\mathrm{Ext}(p) = \{q \ \Delta\text{-type over } D_i \mid q \restriction_E = p, p \subset q, p \ne q\}$. It follows that $\#\{c \mid \Delta\text{-}\mathrm{tp}(c/D_i) = p\} = \#\{c \mid \Delta\text{-}\mathrm{tp}(c/D_i') = p^{p_i}\}$.

With the help of this claim we now change the class of structures we want to avoid and then use induction.

Definition:

- Let $R \in S_k$ $(k > 1)$ and C be the S-structure of the claim. Suppose $1 \le k' \le k$ and $c \in C$, then $R_c^{k'}$ will be a $(k-1)$-ary relation symbol, and the natural interpretation on A will be $R_c^{k'} b_1 \cdots b_{k-1}$ holds iff $R b_1 \cdots b_{k'-1} c b_{k'} \cdots b_{k-1}$ holds in C.

- Let $S' := S \cup \{R_c^{k'} \mid k \in \omega, k > 1, R \in S_k, 1 \le k' \le k, c \in C\}$. Define χ_i' on S' by $\chi_i' \restriction_S = \chi_i$ and $(R_c^{k'})^{\chi_i'} := (R^{\chi_i})^{k'}_{c^{h_i}}$. We can consider A in a natural way as an S'-structure.

- For $T \in \mathfrak{F}$, $\#T \ge 2$, $t \in T$ and $c \in C$ we expand the S-structure $T - \{t\}$ to an S'-structure called T_t^c according to the clause:

$$T_t^c \models R_d^{k'} t_1 \cdots t_{k-1} \Longleftrightarrow d = c \text{ and } T \models R t_1 \cdots t_{k'-1} t t_{k'} \cdots t_{k-1}.$$

- We let $\mathfrak{F}' := \{T \in \mathfrak{F} \mid \#T = 1\} \cup \{T^c_t \mid T \in \mathfrak{F},\ \#T \geq 2, t \in T, c \in C$, for every $R \in S_1$: $T \models Rt \Rightarrow C \models Rc\}$; note that $\mathfrak{F}'$ contains only packed structures.

CLAIM:

(1) *p_i is a χ'_i-permorphism ($1 \leq i \leq n$),*

(2) *$\mathfrak{F}'$ is χ'_i-invariant ($1 \leq i \leq n$).*

(3) *A is $\mathfrak{F}'$-free.*

Proof: (1) follows directly from the properties of the h_i (see (b) of previous claim).

(2) If $T^c_t \in \mathfrak{F}'$ then also $(T^c_t)^{\chi'_i} = (T^{\chi_i})^{c^{h_i}}_t \in \mathfrak{F}'$. For this suppose $R \in S_1$ and $T^{\chi_i} \models Rt$. Now $T^c_t \in \mathfrak{F}'$ and $T \models R^{\chi_i^{-1}} t \Rightarrow C \models R^{\chi_i^{-1}} c \Rightarrow R^{\chi_i^{-1}} x \in \Delta\text{-}\operatorname{tp}(c/D_i) \Rightarrow Rx \in \Delta\text{-}\operatorname{tp}(c^{h_i}/D'_i) \Rightarrow C \models Rc^{h_i}$.

(3) Suppose $T^c_t \in \mathfrak{F}'$ and i: $T^c_t \to_W A$. This leads to a weak homomorphism i': $T \to_W C$, where $i' \restriction_{T - \{t\}} = i$ and $i'(t) = c$. Here we are using that T is irreflexive. This contradicts $C \in \mathfrak{K}_{\mathfrak{F}}$.

By induction we can apply the lemma in this situation and get an $\mathfrak{F}'$-free S'-structure B, $B \supset A$, $f_1, \ldots, f_n \in \operatorname{Sym}(B)$, such that f_i is a χ'_i-permorphism. B satisfies the additional properties mentioned in Lemma 1 for the language S', thus a fortiori for the language S.

So the only thing that still needs checking is that B is $\mathfrak{F}$-free. Again we need that in a certain sense B looks locally like A:

CLAIM:

(1) *If $k > 1$, $R \in S_k$, $1 \leq k' \leq k$, $a \in A$ and $b_1, \ldots, b_{k-1} \in B$, then*

$$Rb_1 \cdots b_{k'-1} a b_{k'} \cdots b_{k-1} \Rightarrow R^{k'}_a b_1 \cdots b_{k-1}.$$

(2) *B is $\mathfrak{F}$-free.*

Proof: (1) By the extra property 2 there exists $f \in \langle f_1, \ldots, f_n \rangle$ such that $a^f \in A$ and $b^f_i \in A$ for all i. Using the extra property 3 let $f = f^{\epsilon_1}_{i_1} \cdots f^{\epsilon_t}_{i_t}$ and $a^{p^{\epsilon_1}_{i_1} \cdots p^{\epsilon_t}_{i_t}} = a^f$ so also $a^{h^{\epsilon_1}_{i_1} \cdots h^{\epsilon_t}_{i_t}} = a^f$. f is a $\chi' = (\chi'_{i_1})^{\epsilon_1} \cdots (\chi'_{i_t})^{\epsilon_t}$-permorphism and $(R^{k'}_a)^{\chi'} = (R^\chi)^{k'}_{a^f}$, where $\chi = \chi' \restriction_S$. Thus $Rb_1 \cdots b_{k'-1} a b_{k'} \cdots b_{k-1} \Rightarrow R^\chi b^f_1 \cdots b^f_{k'-1} a^f b^f_{k'} \cdots b^f_{k-1} \Rightarrow (R^\chi)^{k'}_{a^f} b^f_1 \cdots b^f_{k-1} \Rightarrow (R^{k'}_a)^{\chi'} b^f_1 \cdots b^f_{k-1} \Rightarrow R^{k'}_a b_1 \cdots b_{k-1}$.

(2) We suppose there exist $T \in \mathfrak{F}$ and ρ: $T \to_W B$. Because of the $\mathfrak{F}'$-freeness of B we have $\#T \geq 2$. Pick $t \in T$. We can suppose that $t^\rho \in A$: Otherwise choose $f \in \langle f_1, \ldots, f_n \rangle$ such that $(t^\rho)^f \in A$ and choose $\chi \in \langle \chi_1, \ldots, \chi_n \rangle$ such that f is a

χ-permorphism. Now $\rho \cdot f$ induces a weak homomorphism $T^\chi \to_w B$ with $t^{\rho f} \in A$ and we still have $T^\chi \in \mathfrak{F}$.

Let $t^\rho = a \in A$. We have $T_t^a \in \mathfrak{F}'$. Consider $\rho' = \rho \restriction T - \{t\}$. We want to show that $\rho'\colon T_t^a \to_w B$ to get a contradiction to the $\mathfrak{F}'$-freeness of B. ρ' clearly preserves S-relations so let $R \in S_k, k > 1, 1 \le k' \le k$, $c \in C$ and $t_1, \ldots, t_{k-1} \in T - \{t\}$ such that $T_t^a \models R_c^{k'} t_1 \cdots t_{k-1}$. So $a = c$ and $R t_1 \cdots t_{k'-1} t t_{k'} \cdots t_{k-1}$. Because ρ is a weak homomorphism $R t_1^{\rho'} \cdots t_{k'-1}^{\rho'} a t_{k'}^{\rho'} \cdots t_{k-1}^{\rho'}$. By (1) and $a = c$: $(R_c^{k'}) t_1^{\rho'} \cdots t_{k-1}^{\rho'}$. ∎

One open question is: does (EP) also hold for classes of structures which avoid a given family of packed structures in a stronger sense?

Definition: Let A be an S-structure and $\mathfrak{F}$ a family of packed S-structures. We say A **avoids** $\mathfrak{F}$ (as substructures) iff there does not exist $T \in \mathfrak{F}$ and an embedding $\rho\colon T \hookrightarrow A$.

If $\mathfrak{F}$ is any family of finite packed structures, does (EP) hold for the class of structures avoiding $\mathfrak{F}$?

The only point in the proof where we used weak homomorphism as opposed to embeddings, was the very last claim we proved. If we would be able to find extending structures B which in a stronger sense would locally look like A (i.e. also the reverse of the implication in part (1) of the claim holds), then we would be able to replace weak homomorphisms by embeddings.

Let us give a bound for the structure B satisfying the conclusion in Lemma 6 (or Theorem 5). As in Section 2 it is an iterated exponential function. For this we assume $\mathfrak{F}$ to be finite and we let $c := \#A$ and $t := \sup\{\#T \mid T \in \mathfrak{F}\} \ge 2$ (if $\mathfrak{F}$ is infinite we would need $t := (c+1)$). We suppose the maximal arity of symbols in S is $r \ge 2$ and that S contains l_j many j-ary symbols. We let $p_2(x) = l_1 x + 2 l_2 x^2 + \cdots + r l_r x^r$.

CLAIM: *We can find B with $\#B \le \operatorname{itexp}(2r+t-3, 2p_2(c))$ satisfying the conclusion in Lemma 6.*

We do not give the straightforward but tedious proof here. We only indicate the reason for the different form of the polynomial compared to the formula in Section 2. The notion of Δ-types is different compared to Section 2 and the number of Δ-types over A equals $2^{l_1 + 2l_2 c + \cdots + r l_r c^{r-1}}$. In the construction of the structure C we have to iterate a certain type realizing step c times. We can bound C by $c \cdot (2^{l_1 + \cdots + l_r c^{r-1}})^c = c 2^{p_2(c)}$. The proof of the claim is by induction on t. Note that for $t = 2$ the structure B is automatically $\mathfrak{F}$-free as long as the extra conditions are satisfied. This is because every packed structure of size at most 2 is a link structure.

5. The small index property

In [HHLS] Hodges, Hodkinson, Lascar and Shelah showed how the (EP) for the class of graphs can be used to prove the small index property for the random graph. This proof readily extends to our cases. We do not want to repeat the complete proof here, but for the convenience of the reader we want to indicate how a property like (EP) is useful to understand the automorphism group of the "generic countable structure" of this class. Most of the material up to Theorem 11 is (partly implicitly) contained in [HHLS]. Here we are only looking at the special case of homogeneous ω-categorical structures. Throughout this section S is a finite language.

The idea to look at a generic automorphism can be found in [L] and [T]. The underlying idea of this presentation is based on the introduction in [Hr].

Definition: Let $\mathfrak{K}_{\mathfrak{L}\mathfrak{F}}$ be a class like in Theorem 5. $\mathfrak{K}_{\mathfrak{L}\mathfrak{F}}$ has the amalgamation property (AP). (The (AP) here includes the case —sometimes known as joint embedding property— where the common part is empty.) There exists uniquely up to isomorphism a countable structure M called the **generic structure** for $\mathfrak{K}_{\mathfrak{L}\mathfrak{F}}$ having the following properties:

- Every finitely generated substructure of M belongs to $\mathfrak{K}_{\mathfrak{L}\mathfrak{F}}$.
- For $A \subset B$ finite structures in $\mathfrak{K}_{\mathfrak{L}\mathfrak{F}}$ and $i\colon A \hookrightarrow M$ there exists $j\colon B \hookrightarrow M$ j extending i. Here again we allow $A = \emptyset$.

Such a structure exists for a class $\mathfrak{K}$, if $\mathfrak{K}$ is a class of finite S-structures having the (AP), and which is closed under isomorphisms and substructures. Furthermore M is always homogeneous. See [C] section 2.6 or [Fr], also for the definition of the amalgamation property.

In this paper we always want to assume that the classes we consider are closed under isomorphisms and substructures and we exclude the uninteresting case that there are only finitely many structures up to isomorphism in $\mathfrak{K}$.

If M is a countable structure we equip $G = \mathrm{Aut}(M)$ with the topology of pointwise convergence. G is a polish group, as is G^n equipped with the product topology. So we can talk about comeager sets. See the introduction in [KM] for an introduction to these topological groups.

$(g_1, \ldots, g_n) \in G^n$ is called **locally finite**, if for every $a \in M$ the orbit of a under $\langle g_1, \ldots, g_n \rangle$ is finite. With respect to the automorphism group (EP) says exactly that a generic n-tuple of automorphisms is locally finite (this proposition was probably known to Hrushovski, see the introduction of [Hr]):

PROPOSITION 7: *Let $\mathfrak{K}$ be a class of finite S-structures having (AP). Let M be the generic structure for $\mathfrak{K}$. Then $\mathfrak{K}$ has (EP) iff $\{(g_1, \ldots, g_n) \in \mathrm{Aut}(M)^n \mid (g_1, \ldots, g_n)$ locally finite$\}$ is comeager in $\mathrm{Aut}(M)^n$.*

Proof: "$\Leftarrow$": Suppose that $\{(g_1, \ldots, g_n) \in \mathrm{Aut}(M)^n \mid (g_1, \ldots, g_n)$ locally finite$\}$ is comeager, thus dense in G^n. Take $A \in \mathfrak{K}$ and $p_1, \ldots, p_n$ partial isomorphisms on A. Embedding A into M we can suppose $A \subset M$. The set $\{(g_1, \ldots, g_n) \mid p_i \subset g_i (1 \leq i \leq n)\}$ is by definition of the topology an open set of G^n. It is not empty because of the homogeneity of M. So we find $(g_1, \ldots, g_n)$, $p_i \subset g_i$, such that $(g_1, \ldots, g_n)$ is locally finite. Take B to be the union of all the orbits of elements of A under $\langle g_1, \ldots, g_n \rangle$. Then B solves the extension problem; (EP) holds for $\mathfrak{K}$.

"$\Rightarrow$": We prove that for every $a \in M$ the set $\{(g_1, \ldots, g_n) \mid a^{\langle g_1, \ldots, g_n \rangle}$ finite$\}$ is open and dense. So take $a \in M$ and a non-empty open subset O of G^n. By definition of the topology we find $p_1, \ldots, p_n$ finite partial isomorphisms on M such that $O \supset \{(g_1, \ldots, g_n) \mid p_i \subset g_i (1 \leq i \leq n)\} \neq \emptyset$. Let A be a finite substructure of M containing a and the ranges and domains of the p_i. By applying (EP) we find a finite structure $B \in \mathfrak{K}$, $A \subset B$ and $f_1, \ldots, f_n \in \mathrm{Aut}(B)$, $p_i \subset f_i$. By the defining property of M we can suppose $B \subset M$. By homogeneity of M we can extend the f_i to automorphisms g_i of M. This shows the denseness. For the openness, observe that given $(g_1, \ldots, g_n)$ such that $B = a^{\langle g_1, \ldots, g_n \rangle}$ is finite, all n-tuples of automorphisms which look on B like $(g_1, \ldots, g_n)$ are also in $\{(g_1, \ldots, g_n) \mid a^{\langle g_1, \ldots, g_n \rangle}$ finite$\}$. ∎

In the sequel we want to describe more precisely what in our cases a generic tuple of automorphisms looks like.

Definition: 1. Let $\mathfrak{K}$ be a class of S-structures. By S^{+n} denote $S \cup \{h_1, \ldots, h_n\}$, where $h_1, \ldots, h_n$ are new unary function symbols. By $\mathfrak{K}^{+n}$ we denote the following class of S^{+n}-structures:

$$\{(A, f_1, \ldots, f_n) \mid A \in \mathfrak{K}, f_i \in \mathrm{Aut}(A), A \text{ finite}\}.$$

2. If $\mathfrak{K}^{+n}$ has the (AP) then we define M^{+n} to be the generic countable model of $\mathfrak{K}^{+n}$. $M^{+n} = (M, g_1, \ldots, g_n)$, where g_i is the interpretation of h_i in M^{+n}. By definition of the generic model $(g_1, \ldots, g_n)$ is locally finite.

PROPOSITION 8: *If $\mathfrak{K}$ has the free amalgamation property then so has $\mathfrak{K}^{+n}$.*

PROPOSITION 9:

(a) *Suppose $\mathfrak{K}$ has the free amalgamation property. Denote by M the generic countable structure of this class and by M^{+n} the generic structure of $\mathfrak{K}^{+n}$. Then $M^{+n}\restriction_S \cong M$.*

(b) *Suppose $\mathfrak{K}$ has (AP) and (EP) and $\mathfrak{K}^{+n}$ has (AP). Then $M^{+n}\restriction_S \cong M$.*

(a) This proposition is not as obvious as it might look at first glance. We have to show, that for $A \subset M^{+n}\restriction_S$ and $A \subset B \in \mathfrak{K}$ there exists an embedding $i\colon B \hookrightarrow M$, $i\restriction_A = \mathrm{id}\restriction_A$. We can suppose A to be closed under $\langle g_1,\dots,g_n\rangle$. Let $g_i' := g_i\restriction_A$ for every i. We take $C := B \times \langle g_1',\dots,g_n'\rangle/{\equiv}$, where $(b,g){\equiv}(b^*,g^*)$ iff $b,b^* \in A$ and $b^g = (b^*)^{g^*}$ and embed B into C via $b \mapsto (b,1)/{\equiv}$. For $R\in S$ r-ary and $c_1,\dots,c_r \in C$ we define $Rc_1\cdots c_r$ iff there exists $g \in \langle g_1',\dots,g_n'\rangle$ and $a_1,\dots,a_r \in A$ such that $Ra_1\cdots a_r$ and $c_1 = (a_1,g)$ and $\dots$ and $c_r = (a_r,g)$. We define f_i on C by $((b,g)/{\equiv})^{f_i} := (b,gg_i)/{\equiv}$. Now $(C,f_1,\dots,f_n)$ is in $\mathfrak{K}^{+n}$. For this note that C is nothing else but a free amalgam of (twisted) copies of B over A. Thus there is $j'\colon (C,f_1,\dots,f_n) \hookrightarrow M^{+n}$ with $j'\restriction_A = \mathrm{id}\restriction_A$ and $j = j'\restriction_B\colon B \hookrightarrow M^{+n}\restriction_S$ with $j\restriction_A = \mathrm{id}\restriction_A$. Note the similarity of this relatively easy construction of C and constructions of the form $A\times\Gamma/{\equiv}$ in earlier chapters.

(b) has the same proof. Just take for $(C,f_1,\dots,f_n)$ the structure you get by applying (EP) to B and $g_1\restriction_A,\dots,g_n\restriction_A$. ∎

Side remark: It is a nice and not so easy exercise to prove that this proposition also holds for the class $\mathfrak{K}$ of all finite tournaments. (Use that the automorphism group of a tournament is odd, and that a directed graph, with a group of odd order acting on it, can be completed to a tournament such that the group still acts as automorphisms on this tournament.)

Definition: Let $\mathfrak{K}$ be a class of finite S-structures such that $\mathfrak{K}$ and $\mathfrak{K}^{+n}$ have (AP). Let M be the generic countable model of $\mathfrak{K}$. Let $(g_1,\dots,g_n)$ be a tuple of automorphisms. $(g_1,\dots,g_n)$ is called **Fraïssé-generic** if $(M,g_1,\dots,g_n)$ is isomorphic to the generic model of $\mathfrak{K}^{+n}$.

Note that $\{(g_1,\dots,g_n) \mid (g_1,\dots,g_n)$ is Fraïssé-generic$\}$ is an orbit in $(\mathrm{Aut}\,M)^n$ (under the action of $\mathrm{Aut}M$ on $(\mathrm{Aut}\,M)^n$ by conjugation). The next proposition shows that under certain circumstances it is a comeager orbit (and therefore the only comeager orbit).

PROPOSITION 10 ([HHLS]): *Let $\mathfrak{K}$ be a class of finite S-structures having (AP) and (EP). Suppose further that $\mathfrak{K}^{+n}$ has (AP). Then $\{(g_1,\dots,g_n) \mid (g_1,\dots,g_n)$ is Fraïssé-generic$\}$ is comeager in $\mathrm{Aut}(M)^n$.*

Thus in our cases a generic (in the sense of Baire Category) tuple of automorphisms is Fraïssé-generic.

The proof of the proposition is fairly routine. Transform "$(g_1, \dots, g_n)$ is Fraïssé-generic" into countably many open dense conditions corresponding to the countably many conditions by which the generic structure M^{+n} is defined.

You find this proposition in a different notation and context in [HHLS] (Theorem 2.9). In their terminology the proposition implies that M has "ample generic automorphisms". ∎

THEOREM 11 ([HHLS]): *Let $\mathfrak{K}$ be a class of finite S-structures having (AP) and (EP). Suppose further that $\mathfrak{K}^{+n}$ has (AP) for all n with $1 \leq n < \omega$. Let M be the generic structure of $\mathfrak{K}$. Then M has the small index property.*

Here we say a countable structure M has the **small index property** (SIP), if for every subgroup H of $G = \mathrm{Aut}(M)$, if $[G\colon H] < 2^\omega$ then H is open. See [HHLS] and [KM] for more information about the small index property and sections 4 and 5 of [HHLS] for a proof of the statement "if M has ample generic automorphisms then it has the (SIP)". ∎

Theorem 11 is only implicitly contained in [HHLS], but one can find it more explicitly in a preliminary version of that paper.

COROLLARY 12:

(1) *Let $\mathfrak{F}$ be any set of finite packed irreflexive S-structures, and $\mathfrak{L}$ be a set of irreflexive link-structures. Then $M_{\mathfrak{L}\mathfrak{F}}$, the generic model of $\mathfrak{K}_{\mathfrak{L}\mathfrak{F}}$, has the (SIP).*

(2) *Let $m > 1$. The generic countable K_m-free graph has the (SIP).*

Proof: By Theorem 5 the classes in question have (EP) and by Proposition 4 they have the free amalgamation property, and by Proposition 8 also the corresponding classes $\mathfrak{K}^{+n}$ have (AP). Thus Theorem 11 can be applied. ∎

Here we are especially interested in certain classes of digraphs, which have been introduced by Henson, to prove that there are 2^ω many different ω-categorical theories of directed graphs ([Hen]). Also Peretyat'kin defined 2^ω many different ω-categorical theories in the language consisting of just one binary predicate ([P]), but we are not able to prove the (SIP) for these structures.

Let $S = \{R\}$. We will look at directed graphs, so instead of writing aRb we will also write $a \to b$; directed graphs are S-structures having link type $\mathfrak{L}_2 = \{\cdot, \cdot \to \cdot\}$. A **tournament** is a packed directed graph, i.e. an irreflexive S-structure A, such that for $a, b \in A$ $(a \neq b)$ exactly one of aRb and bRa holds.

The directed graphs Henson introduced are the generic graphs $M_{\mathfrak{L}_2\mathfrak{F}}$ of classes $\mathfrak{K}_{\mathfrak{L}_2\mathfrak{F}}$, where $\mathfrak{F}$ is an infinite family of tournaments. Henson proved that there are 2^ω many non-isomorphic graphs of the form $M_{\mathfrak{L}_2\mathfrak{F}}$.

COROLLARY 13:

(1) *Let $\mathfrak{F}$ be a possibly infinite family of tournaments. Then $\mathfrak{K}_{\mathfrak{L}_2\mathfrak{F}}$ has (EP).*

(2) *The generic structure $M_{\mathfrak{L}_2\mathfrak{F}}$ of the above class has the (SIP).*

(3) *There are 2^ω many non-isomorphic ω-categorical directed graphs having the (SIP).*

(4) *There are 2^ω many non-isomorphic automorphism groups of ω-categorical directed graphs having the (SIP).*

Proof: (1) and (2) are direct consequences of Theorem 5 and Corollary 12, respectively.

(3) follows directly from (2) and the fact that there are 2^ω many non-isomorphic directed graphs of type $\mathfrak{K}_{\mathfrak{L}_2\mathfrak{F}}$, which is proved in [Hen].

(4) follows from (3) if you take into account that two ω-categorical structures have isomorphic automorphism group as topological groups iff they are bi-interpretable in each other (see [AZ]). Use further that in one particular directed graph, there are only countably many different directed graphs interpretable. So the equivalence classes under the equivalence relation "bi-interpretability" on non-isomorphic directed graphs are countable.

Furthermore, observe that for structures M, M' having the (SIP) $\mathrm{Aut}M$ is isomorphic to $\mathrm{Aut}M'$ as a topological group iff they are isomorphic as groups. This is because then the topology can completely be defined from the pure group (not using the operation on M): A subset is open iff it is the union of cosets of subgroups of G having at most countable index. ∎

6. Final remarks

One can hardly expect (EP) to hold for a class $\mathfrak{K}$, if (AP) does not hold for $\mathfrak{K}$. In fact, if you extend a partial isomorphism p: $D \to D'$, $D, D' \subset A \in \mathfrak{K}$, to an automorphism $f \in \mathrm{Aut}(B)$, with $B \in \mathfrak{K}$, $A \subset B$, then you solve the amalgamation problem id: $D \hookrightarrow A$, p: $D \hookrightarrow A$ (via $f\lceil_A$: $A \hookrightarrow B$, id: $A \hookrightarrow B$).

Also the joint embedding property together with (EP) implies (AP): Let $D, B, C \in \mathfrak{K}, D \subset B, D \subset C$. By the joint embedding property there exists $A \in \mathfrak{K}, B \subset A, j$: $C \hookrightarrow A$. Now $j\lceil_D$ is a partial isomorphism on A. By (EP) there exists $E \in \mathfrak{K}$, $A \subset E$ and $f \in \mathrm{Aut}(B)$ extending $j\lceil_D$. Now jf^{-1}: $C \hookrightarrow E$ and $jf^{-1}\lceil_D = \mathrm{id}\lceil_D$. This solves the amalgamation problem.

Having this remark in mind the restriction in Theorem 4 on the family $\mathfrak{F}$ to consist of packed structures looks quite natural. It is the restriction needed to ensure that (AP) holds for the class. In fact we have the following:

LEMMA 14: *Let S be a finite relational language. Let $\mathfrak{F}$ be a set of finite structures. $\mathfrak{K}_{\mathfrak{F}}$ satisfies (AP) iff there exists a set of packed structures $\mathfrak{F}'$ such that $\mathfrak{K}_{\mathfrak{F}} = \mathfrak{K}'_{\mathfrak{F}}$.*

Proof: Let us first remark that $\mathfrak{K}_{\mathfrak{F}}$ has (AP) iff $\mathfrak{K}_{\mathfrak{F}}$ has (fAP). For suppose $\mathfrak{K}_{\mathfrak{F}}$ has (AP). Let $A, B, C \in \mathfrak{K}_{\mathfrak{F}} \cup \{\emptyset\}$ $A \subset B, A \subset C$. By (AP) there exists $D \in \mathfrak{K}_{\mathfrak{F}}$ and embeddings $i_1: B \hookrightarrow D$, $i_2: C \hookrightarrow D$ with $i_1 \restriction_A = i_2 \restriction_A$. But this leads in a natural way to a weak homomorphism which we like to call $i_1 *_A i_2: B *_A C \to_w D$. [In fact if you like to think in terms of the category of S-structures, with weak homomorphisms as morphisms, then $B *_A C$ is the fibered co-product (of $A \hookrightarrow B$ and $A \hookrightarrow C$) in that category, and is therefore uniquely determined up to isomorphism by this property.] But from $B *_A C \to_w D$ and D $\mathfrak{F}$-free follows $B *_A C$ $\mathfrak{F}$-free.

We say $\mathfrak{F}$ is **good**, if for every $T \in \mathfrak{F}$ and every finite S-structure T', if $T \to_w T'$ (by this we mean more formally: if there exists a weak homomorphism $h: T \to_w T'$) then $T' \in \mathfrak{F}$.

CLAIM: *If $\mathfrak{F}$ is good and $\mathfrak{K}_{\mathfrak{F}}$ satisfies (AP), then for every $T \in \mathfrak{F}$ there exists a packed $T' \in \mathfrak{F}$ such that $T' \to_w T$.*

Proof of the Claim: Let $T \in \mathfrak{F}$ be a counterexample minimal in size. As there does not exist a packed $T' \in \mathfrak{F}$ such that $T' \to_w T$, T is not packed. T being not packed means exactly that we can write $T \cong T_1 *_{T_0} T_2$ with $\#T_1 < \#T$ and $\#T_2 < \#T$. As $T \notin \mathfrak{K}_{\mathfrak{F}}$ and as $\mathfrak{K}_{\mathfrak{F}}$ has the (AP), $T_i \notin \mathfrak{K}_{\mathfrak{F}}$ for $i = 1$ or $i = 2$. This means there exists $T' \in \mathfrak{F}$ and $\rho: T' \to_w T_i$. As $\mathfrak{F}$ is good $T_i \in \mathfrak{F}$. But as $T_i \to_w T$ there cannot be a packed structure $T' \in \mathfrak{F}$ such that $T' \to_w T_i$. T_i contradicts the minimality assumption on T.

From the claim follows the Lemma: Let $\mathfrak{F}$ be a family of structures such that $\mathfrak{K}_{\mathfrak{F}}$ has the (AP). Let $\mathfrak{F}'$ be the family (well, formally it is a class) of finite structures T' such that there exists $T \in \mathfrak{F}$ with $T \to_w T'$. We have $\mathfrak{K}_{\mathfrak{F}} = \mathfrak{K}'_{\mathfrak{F}}$. If we let $\mathfrak{F}''$ be the family of elements of $\mathfrak{F}'$ which are packed, then we have $\mathfrak{K}'_{\mathfrak{F}} = \mathfrak{K}_{\mathfrak{F}''}$ by the claim. ∎

There is also a variant of (EP) available in classes of structures, which fail to have the amalgamation property:

Definition: $\mathfrak{K}$ has the **weak extension property** (WEP), if for every finite $A \in \mathfrak{K}$ and partial isomorphisms $p_1, \ldots, p_n$ on A, if there exists a possibly infinite structure $M \in \mathfrak{K}$, $A \subset M$ and $g_1, \ldots, g_n \in \mathrm{Aut}(M)$ such that $p_i \subset g_i$, then there exists a finite structure $B \in \mathfrak{K}$, $A \subset B$ and $f_1, \ldots, f_n \in \mathrm{Aut}(B)$ such that $p_i \subset f_i$.

There is a joint paper with Daniel Lascar ([HL]) where we are going to prove the (WEP) for several classes $\mathfrak{K}$. These properties will be strongly connected to properties of the free groups in finitely many generators. In fact the results in that paper will be stronger than Theorem 5. The methods used there will be quite different from the methods in this paper, and the proof of the stronger theorem will be much longer and more complicated.

Remark: Let us point out that in the context of this paper the weak extension property is not weaker than the extension property:

Let $\mathfrak{K}$ be a class of structures having the amalgamation property. Also suppose that $\mathfrak{K}$ contains the generic countable structure, the "Fraïssé-limit" of its finite structures. Then $\mathfrak{K}$ has the (WEP) iff $\mathfrak{K}$ has the (EP).

Proof: Let $A \in \mathfrak{K}$ be finite, let $p_1, \ldots, p_n$ be partial isomorphisms on A. We have to show that there exists $M \in \mathfrak{K}$, $A \subset M$, $g_1, \ldots, g_n \in \mathrm{Aut}(M)$, $g_i \supset p_i$. Take M to be the generic countable structure and embed A into M. Now p_i extends to $g_i \in \mathrm{Aut}(M)$ because of the homogeneity of M.

In [HL] we prove that for a finite family $\mathfrak{F}$ of structures the class $\mathfrak{K}_{\mathfrak{F}}$ has the (WEP). The condition on the elements of $\mathfrak{F}$ being packed is no longer necessary. In that context it is obvious that working with weak homomorphisms as opposed to embeddings is essential:

Examples:

- The class of total orderings is the class of $\{R\}$-structures of link type $\mathfrak{L}_2 = \{\cdot, \cdot \rightarrow \cdot\}$ (see the example at the beginning of Section 4) which avoids (as substructures) the structures A_2 and C_3, where A_2 consists of 2 points with no edge, and C_3 consists of 3 points a, b, c where $R^{C_3} = \{(a,b),(b,c),(c,a)\}$. As finite orderings allow nontrivial partial isomorphisms but do not allow nontrivial automorphisms and as they are embeddable into the homogeneous ordering of the rationals, this class does not even satisfy (WEP).

- The class of tournaments is the class of $\{R\}$-structures of link type $\mathfrak{L}_2$ which avoid A_2. It is an open question, iff the class of tournaments satisfies (WEP). For this class (as it satisfies (AP)) (WEP) is equivalent to (EP).

The restriction to irreflexive structures in Section 4 and Section 5 is not essential. The theorems also hold without this restriction.

THEOREM 15: *Let $\mathfrak{F}$ be a set of finite packed S-structures. Let $\mathfrak{L}$ be a set of link structures.*

(1) *$\mathfrak{K}_{\mathfrak{L}\mathfrak{F}}$ has (EP).*

(2) *The generic structure $M_{\mathfrak{L}\mathfrak{F}}$ of the class has the (SIP).*

Proof: It suffices to prove (1), as (2) follows from (1) and Section 5. We use a natural method to code arbitrary structures by irreflexive ones: Let R be an r-ary relation symbol and let $x_1, \dots, x_r$ be pairwise distinct variables. For every $r' \leq r$ and every formula $Rx_{i_1} \cdots x_{i_r}$ where $\{i_1, \dots, i_r\} = \{1, \dots, r'\}$ we introduce a new r'-ary relation symbol $R^{(i_1,\dots,i_r)}$ with the intended meaning: $R^{(i_1,\dots,i_r)}x_1 \cdots x_{r'}$ iff $x_1, \dots, x_{r'}$ are pairwise distinct and $Rx_{i_1} \cdots x_{i_r}$. Let S^* be the set of relation symbols obtained in this way. We regard every S-structure A as an irreflexive S^*-structure, which we denote by A^* by defining: $R^{(i_1,\dots,i_r)}a_1 \cdots a_{r'}$ holds in A^* iff $a_1, \dots, a_{r'}$ are pairwise distinct and $Ra_{i_1} \cdots a_{i_r}$ holds.

If A has link type $\mathfrak{L}$ then A^* hast link type $\mathfrak{L}^* = \{L^* \mid L \in \mathfrak{L}\}$. Conversely if B is of link type $\mathfrak{L}^*$ then there exists a unique S-structure A, such that $A^* = B$.

We will suppose that $\mathfrak{F}$ is closed under homomorphic images. By this we mean: If $T \in \mathfrak{F}$ and h: $T \to T'$ is a surjective weak homomorphism, then there exists $T'' \in \mathfrak{F}$ with $T'' \cong T'$. It is no problem to assume $\mathfrak{F}$ to be closed under homomorphic images, as homomorphic images of packed structures are packed, and as for h: $T \to_w T'$: If A is T-free, then it is also T'-free.

We let $\mathfrak{F}^* := \{T^* \mid T \in \mathfrak{F}\}$.

CLAIM: *If A is an S-structure, then A is $\mathfrak{F}$-free iff A^* is $\mathfrak{F}^*$-free.*

The difficult direction is "$\Leftarrow$": Suppose A^* is $\mathfrak{F}^*$-free, but there exist $T \in \mathfrak{F}$ and h: $T \to_w A$. As $\mathfrak{F}$ is closed by homomorphic images we can assume h to be an embedding (we can replace T by the substructure T^h of A). We can now consider h as a mapping between S^*-structures, and we see that h: $T^* \hookrightarrow A^*$. This is a contradiction.

Now we finish the proof of the theorem. We let $A \in \mathfrak{K}_{\mathfrak{L}\mathfrak{F}}$ and $p_1, \dots, p_n$ be partial isomorphisms of A. Then $A^* \in \mathfrak{K}_{\mathfrak{L}^*\mathfrak{F}^*}$ and $p_1, \dots, p_n$ are partial isomorphisms of A^*. By Theorem 5 there exists $C \in \mathfrak{K}_{\mathfrak{L}^*\mathfrak{F}^*}$ $A^* \subset C$ and $f_1, \dots, f_n \in \mathrm{Aut}(C)$ with $p_i \subset f_i$ for every i. Now C is of the form B^*, where $B \in \mathfrak{K}_{\mathfrak{L}\mathfrak{F}}$ and $A \subset B$. ∎

There is a description of the classes $\mathfrak{K}_{\mathfrak{L}\mathfrak{F}}$ in terms of axioms: Let S be a finite relational language. The classes $\mathfrak{K}_{\mathfrak{L}\mathfrak{F}}$ (where $\mathfrak{L}$ is a family of link structures and

$\mathfrak{F}$ is a family of finite packed structures) are the classes which are axiomatizable by axioms of the following forms:

- $\forall x\varphi(x)$, where $\varphi(x)$ is a quantifier free formula (in one variable).
- $\forall\bar{x}(R\bar{x} \rightarrow \varphi(\bar{x}))$, where $r \in \omega$, $\bar{x} = x_1 \cdots x_r$, R is r-ary and $\varphi(\bar{x})$ is a quantifier free formula.
- $\neg\exists\bar{y}\bigwedge_{i=1}^{t}\varphi_i(\bar{y})$, where $k, t \in \omega$, $\bar{y} = y_1 \cdots y_k$, each $\varphi_i(\bar{y})$ is an atomic formula, for every j, j' with $1 \leq j < j' \leq k$ there exists i $(1 \leq i \leq t)$ such that y_j and $y_{j'}$ both appear in $\varphi_i(\bar{y})$.

It is also no problem to allow finitely many constants in the language as long as one insists that all the domains of the partial isomorphisms in question contain all the constants. For simplicity we will not speak about structures of a given link type.

COROLLARY 16: *Let S be a finite relational language with constants. Let $\mathfrak{F}$ be a family of finite packed structures each in a sublanguage of S. Let $\mathfrak{K}_{\mathfrak{F}}$ be the class of $\mathfrak{F}$-free S-structures. $\mathfrak{K}_{\mathfrak{F}}$ has the (EP). Hereby a partial isomorphism p on A is an isomorphism between two S-substructures of A.*

The condition on the partial isomorphisms just means that for all the constants c in the language and every one of the partial isomorphisms p, c (or rather the interpretation c^A of c in A) is in $D(p)$ and $c^p = c$.

Proof: We replace in the language S every constant c by a unary predicate U_c, which is intended to hold only for the interpretation of c. We denote this new relational language by S^*. Every structure A in a sublanguage of S can be considered in a natural way as an S^*-structure, which we denote by A^*. Hereby we interpret relation symbols from S which have not been in the language of A by the empty set, and the same for symbols U_c, where c has not been in the language of A. We let $\mathfrak{F}^* = \{T^* \mid T \in \mathfrak{F}\}$.

Now let $A \in \mathfrak{K}_{\mathfrak{F}}$ be finite and $p_1, \ldots, p_n$ be partial isomorphisms on A. We have $A^* \in \mathfrak{K}_{\mathfrak{F}^*}$. By Theorem 15 there exists a finite $E \in \mathfrak{K}_{\mathfrak{F}^*}$ and $f_1, \ldots, f_n \in \mathrm{Aut}(E)$, such that $A^* \subset E$ and $p_i \subset f_i$ for each i. We only have to check that $U_c^E = U_c^{A^*} = \{c\}$ to conclude that $E = B^*$ for some $B \in \mathfrak{K}_{\mathfrak{F}}$ and to finish the proof.

But we can easily suppose that for every element $d \in E$ there exists $f \in \langle f_1, \ldots, f_n\rangle$ such that $d^f \in A^*$. Now if we take $d \in U_c^E$ and suppose that $d^f \in A$, then $d^f \in U_c^A$ and therefore $d^f = c$. But for every $f' \in \langle f_1, \ldots, f_n\rangle$ $c^{f'} = c$ as c is fixed by every f_i. In particular $d = c^{f^{-1}} = c$. ∎

Reference

[AHN] H. Andréka, I. Hodkinson and I. Németi, *Finite algebras of relations are representable on finite sets*, to appear in Journal of Symbolic Logic.

[AZ] E. Ahlbrandt and M. Ziegler, *Quasi finitely axiomatizable totally categorical theories*, Annals of Pure and Applied Logic **30** (1986), 63–82.

[C] P. Cameron, *Oligomorphic Permutation Groups*, London Mathematical Society Lecture Notes Series 152, Cambridge University Press, 1990.

[Fr] R. Fraïssé, *Sur l'extension aux rélations de quelques propriété des ordres*, Annales Scientifiques de l'École Normale Supérieure **71** (1954), 361–388.

[Gr] M. Grohe, *Arity hierarchies*, Annals of Pure and Applied Logic **82** (1996), 103–163.

[GGK] M. Goldstern, R. Grossberg and M. Kojman, *Infinite homogeneous bipartite graphs with unequal sides*, Discrete Mathematics **149** (1996), 69–82.

[Hen] W. Henson, *Countable homogeneous relational structures and $\aleph_0$-categorical theories*, The Journal of Symbolic Logic **37** (1972), 494–500.

[Hg] B. Herwig, *Extending partial isomorphisms*, Combinatorica **15** (1995), 365–371.

[HL] B. Herwig and D. Lascar, *Extending partial isomorphisms and the profinite topology on the free groups*, to appear in Transactions of the American Mathematical Society.

[Hr] E. Hrushovski, *Extending partial isomorphisms of graphs*, Combinatorica **12** (1992), 411–416.

[HHLS] W. Hodges, I. Hodkinson, D. Lascar and S. Shelah, *The small index property for ω-stable, ω-categorical structures and for the random graph*, Journal of the London Mathematical Society **48** (1993), 204–218.

[KM] R. Kaye and D. Macpherson, *Automorphisms of First-Order Structures*, Clarendon Press, Oxford, 1994.

[L] D. Lascar, *Autour de la propriété du petit indice*, Proceedings of the London Mathematical Society **62** (1991), 25–53.

[P] M. G. Peretyat'kin, *On countable theories with a finite number of denumerable models*, Algebra i Logika **12** (1973), 570–576 (310–326 in the English translation).

[T] J. K. Truss, *Generic automorphisms of homogeneous structures*, Proceedings of the London Mathematical Society **65** (1992), 121–141.